THE ELEVENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Eleventh Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in St. Louis, Mo., on Friday and Saturday, September 16–17, 1904. The sessions opened at 10 A. M. and 2 P. M. on each day in Room 1 of the Library building of Washington University, known in connection with the Louisiana Purchase Exposition as the Hall of Congresses.

The following thirty-eight members of the Society were in attendance:

Dr. L. D. Ames, Professor H. Y. Benedict, Professor G. A. Bliss, Professor Maxime Bôcher, Professor A. S. Chessin, Dr. Saul Epsteen, Mr. E. B. Escott, Professor Fanny C. Gates, Professor G. W. Greenwood, Director J. G. Hagen, Professor G. B. Halsted, Professor Harris Hancock, Professor M. W. Haskell, Professor E. R. Hedrick, Dr. E. V. Huntington, Professor J. I. Hutchinson, Dr. Edward Kasner, Dr. H. G. Keppel, Professor James McMahon, Professor H. P. Manning, Professor J. L. Markley, Professor Heinrich Maschke, Professor E. H. Moore, Professor Simon Newcomb, Professor M. B. Porter, Mr. W. H. Roever, Mr. Oscar Schmiedel, Miss I. M. Schottenfels, Professor J. B. Shaw, Professor E. B. Skinner, Professor H. E. Slaught, Professor W. B. Smith, Professor Ormond Stone, Professor J. H. Tanner, Mr. E. H. Taylor, Professor H. S. White, Professor B. F. Yanney, Dr. J. W. Young, Professor Alexander Ziwet.

Professor Henri Poincaré, of Paris, and Professor Gino Fano, of Turin, were also present by special invitation.

In the absence of the respective officers of the Society, Professor Alexander Ziwet was elected chairman and Professor M. W. Haskell secretary of the meeting. The Council announced the election of the following persons to membership in the Society: Dr. H. A. Converse, Johns Hopkins University; Mr. E. L. Dodd, University of Iowa; Dr. Saul Epsteen, University of Chicago; Professor Tullio Levi-Civita, University of Padua; Professor J. C. Lymer, Lawrence University, Appleton, Wis; Professor W. F. Moncreiff, Winthrop College,

S. C.; Dr. Clara E. Smith, Yale University; Professor D. T. Wilson, Case School of Applied Science. Fourteen applications for membership in the Society were received.

A committee, consisting of the President of the Society, Dr. McClintock and Professors E. H. Moore, Stringham and Tyler, was appointed to prepare and report at the October meeting a list of nominations of officers and other members of the Council to be voted for at the annual election in December.

On Saturday afternoon Professor Poincaré accepted an invitation to address the Society and gave an account of certain new theorems on the existence of closed geodesics upon closed convex surfaces. During this session also the chairman introduced to the Society Professor Gino Fano, of the University of Turin, who brought greetings from Italian mathematicians.

At the close of Friday's session an excursion was made to the palace of education, where Professor Hedrick explained the exhibit of the University of Missouri, in particular certain mathematical models constructed by advanced students in his department last year. Of marked interest was a model made by Mr. Ingold to accompany his paper (12) of the list below; this gave in red and blue wire a large number of lines representing real and imaginary points of a real circle. Other models related to geodesic lines, subgroups of the modular group, and analysis situs.

It is to be regretted that the invitation of the committee, to forward early synopses of papers intended for the meeting, was so largely disregarded. Notwithstanding this, debate was in parts of the programme unusually lively, pedagogical questions naturally arousing most discussion.

The following papers were presented at this meeting:

(1) Dr. L. D. AMES: "Supplementary communication on the division of space by a closed surface."

(2) Professor G. A. BLISS: "On the problem of the calculus of variations involving several unknown functions."

(3) Dr. D. R. CURTISS: "Sur certains théorèmes de la valeur moyenne."

(4) Dr. D. R. CURTISS: "Sur la théorie des fonctions hypergéométriques."

(5) Mr. E. B. ESCOTT: "The expression of a quadratic surd as a continued fraction."

(6) Professor MAXIME BÔCHER: "On certain pseudo-mathematical or logical paradoxes."

(7) Dr. W. B. FITE: "The successive commutators of a group."

(8) M. MAURICE FRÉCHET: "Sur les opérations linéaires."

(9) Professor E. R. HEDRICK: "The necessary and sufficient conditions for the inverse problem of the calculus of variations."

(10) Professor J. B. SHAW: "Composition of a linear associative algebra."

(11) Dr. E. V. HUNTINGTON: "A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups."

(12) Mr. LOUIS INGOLD: "Real representation of the real and imaginary portions of a plane locus."

(13) Dr. F. S. MACAULAY: "On a method of dealing with intersections of plane curves."

(14) Professor G. A. MILLER: "Note on Burnside's Theory of Groups."

(15) Miss I. M. SCHOTTENFELS: "On a set of generators for certain substitution and Galois field groups."

(16) Professor E. R. HEDRICK: "The differential notation."

(17) Dr. OSWALD VEBLEN: "The fundamental theorem of analysis situs."

(18) Professor HENRI POINCARÉ: "Closed geodesics on a closed convex surface."

(19) Professor H. S. WHITE: "Certain quartic and quintic surfaces admitting infinitesimal collineations."

(20) Dr. J. W. YOUNG: "On the use of hypercomplex numbers in certain problems of the modular group."

(21) Mr. H. S. VANDIVER: "On reduction algorithms for the solution of the linear equation in a finite field."

(22) Professor T. J. I'A. BROMWICH: "The classification of quadrics."

(23) Professor L. E. DICKSON: "Explicit exhibition of all the subgroups of orders the three highest powers of p in the group G of all m-ary linear homogeneous transformations modulo p."

(24) Professor L. E. DICKSON: "Determination of all the subgroups of the group of all binary linear homogeneous transformations of determinant unity in the $GF[p^n]$."

M. Fréchet's paper was communicated to the Society through Professor E. H. Moore. Mr. Vandiver was introduced by Professor Crawley. In the absence of the authors, Dr. Cur-

tiss's papers were read by Professor Bôcher, Dr. Fite's paper by Professor Hutchinson, M. Fréchet's and Dr. Macaulay's papers by Professor Moore, Mr. Ingold's paper by Professor Hedrick, Dr. Veblen's paper by Professor Bliss, and the papers of Professor Miller, Professor Bromwich, and Professor Dickson were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the BULLETIN for March, 1904, Dr. Ames published a communication on the theorem of analysis situs relating to the division of the plane or of space by a closed curve or surface. While no fundamental changes have been made in the methods sketched in that paper, certain minor alterations have beenf ound advisable in completing the details of the proof. In particular, the definition of a solid angle has been made to depend upon a certain surface integral, in place of the tentative definition given in the previous article. This, and a few other unimportant changes are explained in the present paper, which is to appear together with the previous paper, in extended form, in the *American Journal of Mathematics*.

2. The paper read by Professor Bliss concerned the problem of the calculus of variations corresponding to the integral

$$I = \int f(x;\ y_1, y_2, \cdots, y_n;\ y_1', y_2', \cdots, y_n')dx.$$

The necessary conditions of Legendre and Jacobi for this problem have been treated by many writers, but usually in a very complicated fashion. Professor Bliss endeavored to establish these as simply as possible by making use of the existence theorems for a set of linear differential equations, and by an extension of a method used by Schwarz for $n = 1$. In addition he gave a proof of the condition (Weierstrass) which becomes necessary if the minimum is to be a so-called *strong* one.

The requirements for a permanent sign of the second variation have also been the subject of many memoirs, but the sufficient conditions for a minimum in the Weierstrassian sense, in particular for a strong minimum, have been discussed only incompletely.* In Professor Bliss's paper, the construction of a field was explained and these conditions derived.

*See Encyklopädie der mathematischen Wissenschaften, II A 8, p. 608 and II A 8a, p. 633.

3. The importance of certain theorems of mean value for functions of a real variable gives in itself an interest to analogous propositions for a complex variable. A number of theorems of this sort are presented in Dr. Curtiss's paper. To give an example of the form these assume, it is found that the familiar equation

$$f(z) - f(a) = (z - a)f'(a + \theta(z - a))$$

has, when $f(z)$ is a function of the complex variable analytic at a, a solution θ within the circle defined by the inequality $|\theta - \frac{1}{2}| < \frac{1}{2}$, provided z is sufficiently near a. This last condition is shown, by illustrations, to be essential. Similar theorems for integrals are developed. This paper will appear in the *Bulletin des Sciences mathématiques.*

4. Dr. Curtiss's second paper, which will appear in the *Annales de l'Ecole normale supérieure,* is a resumé of those portions of his thesis (published in the *Memoirs of the American Academy of Arts and Sciences,* volume 13, No. 1 (1904)) which are devoted to hypergeometric functions without apparently singular points. It is well known that Riemann's celebrated memoir omitted from consideration certain cases, some explicitly, others implicitly. Following Riemann's procedure, a new definition which includes all cases is taken as the point of departure and the chief properties of function families thus defined are developed, the author indicating what changes and additions must be made in order to apply Riemann's methods to the larger problem.

5. Mr. Escott's paper is a continuation of that on the "Expression of the square root as a continued fraction," presented at the April meeting of the Chicago Section of the Society. Any infinite continued fraction which contains a repeating cycle of partial quotients is equivalent to a number which is the root of a quadratic equation. If the number is the root of a pure quadratic equation, *i. e.,* a square root, the continued fraction is of the form $(a: b^*, c, d, \cdots, d, c, b, 2a^*)$, where the non-repeating part consists of a single quotient and the cycle consists of quotients symmetrically arranged with the exception of the last quotient, which is double the first quotient. The stars indicate the cycle. In this paper some of the problems con-

sidered are the determination of the form of the number N when the cycle of partial quotients is given, the solution of Fermat's equations $x^2 - Ny^2 = \pm 1$ (commonly called the pellian equations), also the solution in odd numbers of the related equation $x^2 - Ny^2 = 4$ which arises in studying the relative numbers of properly and improperly primitive classes of reduced forms of determinant N.

6. Professor Bôcher's paper began with a discussion of Russell's paradox concerning the class of all classes no one of which contains itself as an element. It was pointed out that this paradox would seem to invalidate Russell's theory of cardinal numbers in a point which that writer regards as essential. Another paradox was then discussed which is related to a remark attributed by Richard in his book, Sur la philosophie des mathématiques, to Tannery, according to which there exist infinite decimal fractions whose figures proceed according to no law.

7. Dr. Fite defines a second commutator of a group as a commutator formed by associating a commutator with any operator of the group. In general, an ith commutator is one formed by associating an $(i-1)$th commutator with any operator of the group. The ith commutator subgroup is the subgroup generated by the ith commutators. A necessary and sufficient condition that a group be isomorphic with a group of class i is derived. Several relations connecting the order and class of the various commutator subgroups and the class of the original group are derived. Finally, the notion of the successive commutators is made use of to determine a limit to the class of a group of order p^a (p a prime) that contains two subgroups of order p^{a-1} and of given classes.

8. Pincherle has given a development into series for a linear operator U_f applied to a considerably restricted function f. In the present note M. Fréchet, making use of the Fourier series for f, is able to obtain a development of U_f into a convergent series if f satisfies Dirichlet's conditions. A development into a series possessing a generalized limit (in the sense of Cesàro) may be given in case f is *any* continuous function. These results are applied to the discussion of such linear operators as

$$Y_f = \int_0^\pi f(y)K(y)dy,$$

where the function $K(y)$ is integrable in Riemann's sense or in the extended sense explained by Lebesgue in his recent Leçons sur l'intégration.

9. Professor Hedrick's first paper is in abstract as follows: The direct problem of the calculus of variations has never been reduced to the application of a single condition which is at once necessary and sufficient. In the inverse problem, given a two parameter family of curves which satisfy the Jacobi condition inside a certain region, integrals may be found which satisfy the Lagrange differential equation. In order that any one curve of the given family should render the integral selected a minimum, it is sufficient, by Weierstrass's condition, that the second derivative of the integrand with respect to y' should be positive for any point (x, y) on the curve and for any value of y' whatever. But since a curve of the given family passes through each point of the region in any preassigned direction, the necessary condition that every one of the curves of the family render the integral a minimum, by Legendre's condition, coincides with the sufficient condition stated above. Hence that condition, for any point (x, y) inside the region, and for any value of y' whatever, is both necessary and sufficient for the inverse problem.

10. It is shown in Professor Shaw's paper that every linear associative algebra consists of sub-algebras $A_{ij}\,(i, j = 1, 2, \cdots, s)$. Of these each A_{ii} consists of "direct" units and is of the form $\{Q \cdot P\}$ where Q is a quadrate of order κ_i^2, and P a non-quaternionic algebra of order ρ_i, A_{ii} being of order $\kappa_i^2\rho_i$. The algebras $A_{ij}\,(i \neq j)$ consist of units whose squares vanish, such that if the product of two units α_{ij}, α_{jk} from two such algebras A_{ij}, A_{jk} does not vanish then $\alpha_{ij}\alpha_{jk} = c_{ik}\alpha_{ik}$, where c_{ik} is a scalar. Also

$$\alpha_{ij}\alpha_{kl} = 0, \qquad (j \neq k),$$

$$\alpha_{ij}\alpha_{jj} = \alpha_{ii}\alpha_{ij} = \alpha_{ij}.$$

Certain theorems are given as consequences of these general theorems. Some of the sub-algebras $A_{ij}\,(i \neq j)$ may be absent, so that there are s^2, or less than s^2, such sub-algebras.

11. The postulates for real algebra given in Dr. Hunting-

ton's paper may be analyzed into three groups: (1) propositions concerning the relation $<$, which, taken by themselves, form a set of independent postulates for the theory of a one-dimensional continuum or a continuous scale; (2) propositions concerning the operation $+$, which, taken by themselves, form a set of independent postulates for the theory of groups; and (3) a proposition connecting the two symbols $<$ and $+$.

All these postulates are shown to be independent, that is, the list contains no redundancies; and the system which they determine is shown to be unique, that is, there is essentially only one system (namely, the system of all real numbers) in which a relation $<$ and an operation $+$ are so defined as to satisfy all the postulates. The existence of the system of real numbers proves, moreover, the consistency of the postulates.

The postulates for a one-dimensional continuum are the obvious ones (see, for example Schoenflies's "Bericht über Mengenlehre"),* but their independence has not heretofore been established.

The postulates for the theory of groups carry the analysis farther, it is thought, than the earlier sets† given by the writer and by E. H. Moore.

The list as a whole is more satisfactory than the writer's earlier set of postulates for real algebra‡ in several respects; first, the separation of the postulates concerning $<$ from those concerning $+$ is now complete; secondly, the individual postulates are more nearly simple statements (and are hence more numerous); and finally, no assumption is now made in regard to the existence of any kind of numbers. This last improvement was suggested by a recent paper of Burali-Forti's,§ to which special attention is directed.

Dr. Huntington's paper was presented also to the St. Louis Congress of arts and sciences in the section of algebra and analysis; it will be printed in full in the *Transactions* of the Society.

12. Many representations of the imaginary points of a plane locus have been given, the Riemann surface representation

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* *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 8 (1900).

† See *Transactions*, vol. 3 (1902), pp. 485–492, and vol. 4 (1903), pp. 27–30.

‡ Ibid., vol. 4 (1903), pp. 358–370.

§ *Atti della R. Accademia delle Scienze di Torino*, vol. 39 (1904).

being among the best known. It is desirable, however, to have a representation which will more satisfactorily visualize the locus. Ordinary point space being three-dimensional, it is obvious that some other space element must be chosen as the basis of the geometric representation, if the whole picture is to be given in a single ordinary space. A representation is here suggested in which the element taken is a straight line. Since line geometry in space is four-dimensional, this choice is a very natural one. Making a suitable convention regarding the correspondence of number pairs and straight lines, it turns out that the simpler loci are represented by simple configurations, the ordinary appearance of the real portion of the locus being retained.

The coordinate planes may be looked upon as complex fields analogous to the ordinary complex plane, and the relations usually developed with the aid of the complex plane, including the Riemann surface relations, can be reproduced in this new representation.

Models of simple loci may be readily constructed in which the imaginary as well as the real points of the locus are apparent to the eye, and relations which are troublesome in the ordinary representation become self-explanatory. Such a model has been constructed for the intersection of straight lines with a circle and was on exhibition in the Educational building at St. Louis.

13. The paper of Dr. Macaulay is a new investigation of certain theorems which he gave in the *Proceedings of the London Mathematical Society*, volume 31 (1899), pages 381–423, and which were later discussed in a simpler manner by Professor Charlotte A. Scott in the *Transactions*.

Noether's fundamental theorem states that certain conditions are sufficient in order that a polynomial F in two variables x, y may be of the form $Af + B\phi$, where f, ϕ are given polynomials, and A, B undetermined polynomials. These conditions may be thus expressed: Change the origin to any point a, b, *i. e.*, substitute $x + a$, $y + b$ for x, y in F, f, ϕ, and expand in ascending powers of x, y; then, whatever finite values a, b may have, F must be of the form $A'f + B'\phi$, where A', B' are undetermined infinite power series, which are entirely different for different origins. Any origin at a point of intersection of the curves f, ϕ supplies a certain number of conditional equa-

tions for the coefficients of F; the conditions which different origins supply are quite independent of one another, and there is no limit to the number of conditions which a single origin may supply. On the other hand, any origin which is not a point of intersection of f, ϕ supplies no conditions; for if f (say) does not vanish at the origin, F/f is a power series, so that F is of the form $A'f$, and therefore of the form $A'f + B'\phi$.

The theorems in question relate to the conditions which are supplied for F in the most general case when the origin is a point at which the curves f, ϕ have multiple points of any order and complexity, and contact of any kind.

14. In Professor Miller's review of Burnside's Theory of Groups (BULLETIN, volume 6 (1900), page 390) an error in regard to a fundamental theorem seems to have been noted too briefly. The theorem relates to the possible types of subgroups in an abelian group and is developed by Burnside in articles 45–47. The first two articles seem free from error, but article 47 begins with the false assumption that the condition

$$\sum_1^t n_u \leq \sum_1^t m_u \qquad (t = 1, 2, \cdots, r),$$

which has only been proved as *necessary* for the existence of a subgroup of type $(n_1, n_2, \cdots, n_s)$, is also sufficient. Hence the developments of this article and the part of Theorem III which depends upon them are unreliable. It is not difficult to see from the results which Burnside gives in the earlier articles that the necessary and sufficient condition for the existence of the subgroup in question is

$$n_t \leq m_t \qquad (t = 1, 2, \cdots, s).$$

The fundamental importance of this theorem seems to demand an explicit statement in regard to this error even if it could escape the notice only of the beginner.

15. Miss Schottenfels's paper determines that the following substitutions

$$C_h\colon\ x_i' = x_{i+1},\ x_k' = x_1 + hx_2 \quad (i = 1, 2, \cdots, k-1;\ h = 0, 1)$$

are generators for (1) all N-ary linear substitution groups of determinant unity, with integral coefficients, (2) all linear and linear fractional groups in the Galois field $[2^1]$.

16. While the differential notation is now presented in an unobjectionable form in many books on the calculus, there still remains a tendency, either on the part of the student or on the part of the author, to depart from the correct definitions immediately after they are given. In Professor Hedrick's second paper the possibility of substituting a consistent and simple derivative notation in the place of the ordinary differential notation is insisted upon, and the advantages of doing so are pointed out. It is shown that the advantages of symmetry, compactness, and clearness may be retained, and that any proof can be followed as readily as in the differential form. The notation suggested involves the introduction of variable parameters in terms of which the variables are expressed.

17. Dr. Veblen's paper contains a proof of Jordan's theorem that a simple closed curve decomposes a plane in which it lies into two regions. The definition of the curve and the proof of the theorem do not involve any metrical hypotheses, *i. e.*, there is no reference to coördinates or to axioms of congruence. If P is any point not on a curve c, then any point C of c which can be joined to P by a broken line not meeting c except in C is said to be *finitely accessible* from P. That the points of c finitely accessible from P are everywhere dense on c is one of the principal lemmas. The paper is to be published in the *Transactions*.

18. Considering an everywhere convex closed surface, Professor Poincaré first observed that the closed geodesics upon such a surface might be divided into two classes according as the number of double points of the geodesics were even or odd, since under a continuous deformation of the surface such double points could be made to disappear, not singly, but only in pairs.

The determination of these closed geodesics depends on the determination of the maxima, minima and minimaxima of a certain function; in general, therefore, the total number of such geodesics will be an odd number and there must always exist at least one. Professor Poincaré stated as his belief that there are always at least three.

To look at the subject from another point of view, we might ask what is the shortest closed curve on a closed convex surface. This would of course in general be infinitesimal, but not if the curve is to be a geodesic. The (gaussian) total curva-

ture of a closed convex surface is 4π. If the surface be separated into two parts by a closed curve, it may happen that the total curvature of each part is 2π; that this may be the case, the curve must be a geodesic. The converse is true for closed geodesics without double points, and also under suitable modification for geodesics with double points.

The investigation is of importance in connection with the discussion of trajectories in celestial mechanics.

19. Picard's identical differential equation for a quartic surface possessing a linear exact differential of the first kind is

$$\theta_1 \frac{\partial f}{\partial x_1} + \theta_2 \frac{\partial f}{\partial x_2} + \theta_3 \frac{\partial f}{\partial x_3} + \theta_4 \frac{\partial f}{\partial x_4} \equiv 0.$$

Professor White interpreted this geometrically as stating that the quartic $f = 0$ is invariant under the infinitesimal collineation $x_i = y_i + \lambda \cdot \theta_i (i = 1, 2, 3, 4)$. This interpretation leads by inspection method to Poincaré's two quartics of the kind described. The class of quartics and quintics admitting such collineations of the type (1 1 1 1) is examined in some detail. (Much of this work has no doubt been anticipated by Professor H. B. Newson, whose paper, presented at the ninth summer meeting of the Society has not yet been published.)

20. By the modular group Γ is meant the totality of substitutions

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where $\alpha, \beta, \gamma, \delta$ are integers such that $\alpha\delta - \beta\gamma = 1$. If the substitutions of Γ are transformed by the substitution

$$W(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are numbers of a commutative hypercomplex number system E such that $ad - bc$ is neither zero nor a divisor of zero, a group Γ' simply isomorphic with Γ is obtained whose coefficients are numbers of E. Such a hypercomplex representation of Γ may be used to study the sub-

groups of Γ. If E be the two-unit system whose units e_0 and e_1 satisfy the relations $e_0^2 = e_0$, $e_0 e_1 = e_1 e_0 = e_1$, $e_1^2 = 0$, the substitutions of Γ' in which all the coefficients of e_1 are zero clearly form a group. The subgroups thus defined by various devices W are all cyclical; and in fact all cyclical subgroups of Γ may be defined in this way. Further, the totality of substitutions in Γ' in which all the coefficients of e_1 are integers congruent to zero mod. n (n any integer) form a subgroup. The subgroups obtained in this way are all congruence groups. For different W's Dr. Young finds that it is possible to obtain very simply an arithmetic definition of a large series of the congruence subgroups of Γ in a form apparently never before exhibited.

21. Since any finite field can be represented as a Galois field, the arithmetic significance of the latter is considerable. Mr. Vandiver's paper gives a new algorithm (in reduction form) for the solution of the linear equation in such a field. It has the peculiarity that the same moduli are retained throughout the reduction, and is simpler in theory than the well known continued fraction method, as well as entirely different in character.

In reference to the other fields the residue classes with respect to an ideal prime in an algebraic field are considered in connection with a linear congruence.

22. Mr. J. L. Coolidge has recently given a classification of quadrics in hyperbolic space (*Transactions*, volume 4, page 161), basing his work on Clebsch's discussion of the mutual relations of two quadrics.

Professor Bromwich examines the same problem with the help of the Sylvester-Weierstrass classification of quadratic forms, using invariant factors. It appears that certain of Coolidge's reduced forms can be still further simplified, as they contain coefficients which are not invariants of the two quadrics. The types of quadrics in elliptic and parabolic spaces are then briefly discussed, to show the features in which they agree with those in hyperbolic space; there is also prefixed to the paper a short summary of those results relating to the reduction of quadratic forms which are used in the subsequent work.

23. The first paper by Professor Dickson exhibits all the subgroups of order p^μ, $p^{\mu-1}$, $p^{\mu-2}$ of G, where $\mu \equiv \frac{1}{2}m(m-1)$

is the exponent of the highest power of p dividing the order of G. A fortunate choice of notation gives the results and proofs a more compact and luminous form than is usual in so general an investigation in group theory. Repeated use is made of the theory of commutator subgroups. The paper gives a wide generalization, by different methods of proof, of the results in the author's article in the BULLETIN for May, 1904.

24. The second paper by Professor Dickson forms a preliminary chapter in his investigation, as research assistant to the Carnegie Institution of Washington, on the resolvents for the p-section of the periods of hyperelliptic functions of $2m$ periods. The group for this p-section contains subgroups isomorphic with the binary group Γ of determinant unity. Having determined all the subgroups of Γ, we derive at once the subgroups of the quotient group of linear fractional transformations. As the latter are required for the elliptic modular theory, the present procedure is an instance of mathematical economy.

M. W. HASKELL,
H. S. WHITE.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

THE sixth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, October 1, 1904, at the University of California. The following fifteen members were present:

Dr. E. M. Blake, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor A. O. Leuschner, Dr. J. H. McDonald, Professor G. A. Miller, Professor H. C. Moreno, Professor C. A. Noble, Dr. T. M. Putnam, Professor Irving Stringham, Professor A. W. Whitney.

Major P. A. MacMahon presided during the morning session and Professor Stringham during the afternoon session. During the morning session the following officers were elected for

the ensuing year: Professor M. W. Haskell, chairman; Professor G. A. Miller, secretary; Professors M. W. Haskell, D. N. Lehmer, G. A. Miller, programme committee.

The following papers were read at this meeting:

(1) Professor E. J. WILCZYNSKI: "General projective theory of space curves."

(2) Professor L. M. HOSKINS: "Stresses in an elastic sphere due to a superficial layer of heavy matter of uniform thickness bounded by a circle."

(3) Dr. T. M. PUTNAM: "Concerning the factors of $(p^2-1)^2$ that are of the form $px+1$, and allied problems."

(4) Major P. A. MACMAHON: "Groups of linear differential operators."

(5) Professor H. F. BLICHFELDT: "On primitive continuous groups."

(6) Professor D. N. LEHMER: "Figures invariant in space of three dimensions under the most general projective transformation."

(7) Professor G. A. MILLER: "Determination of all the groups of order 2^m which contain an odd number of cyclic subgroups of composite order."

The programme provided also for a "conference on recent investigations in the foundations of geometry." This conference was opened by Professor Stringham, who was followed by Dr. J. H. McDonald. A number of the high school teachers of mathematics took part in the discussion, which had reference mainly to the influence of the recent investigations on the teaching of elementary geometry.

Major MacMahon was introduced by Professor Stringham. In the absence of the author, Professor Wilczynski's paper was read by Professor Haskell. Abstracts of the papers follow below: The abstracts are numbered to correspond to the titles in the list above.

1. Professor Wilczynski considers a linear differential equation

$$y'''' + 4p_1y''' + 6p_2y'' + 4p_3y' + p_4y = 0, \tag{1}$$

together with its invariants and covariants under the transformation

$$y = \lambda(x)y, \quad \xi = \xi(x), \tag{2}$$

λ and ξ being arbitrary functions. The covariants are functions of the semicovariants y and

$$(3) \qquad \begin{aligned} z &= y' + p_1 y, \quad \rho = y'' + 2p_1 y' + p_2 y, \\ \sigma &= y''' + 3p_1 y'' + 3p_2 y' + p_3 y. \end{aligned}$$

If $y_1, \cdots, y_4$ form a fundamental system of (1), these functions y_k may be taken as homogeneous coördinates of a point P_y on a curve C_y, the integral curve of (1). By means of (3), three other curves C_z, C_ρ, C_σ are obtained, which are closely connected with C_y. These curves are denoted as the derivatives of C_y with respect to x of the first, second and third kind respectively. If the variable x be changed, all of these curves except C_y itself are transformed into others. The paper is concerned with these curves and with the ruled surfaces generated by the edges of the tetrahedron $P_y P_z P_\rho P_\sigma$, each of which, corresponding to all of the transformations of the independent variable, depends upon an arbitrary function.

The author succeeds in defining these curves geometrically by introducing the notions osculating plane, conic, cubic and linear complex of the curve C_y. Among the surfaces generated by $P_y P_\sigma$ there exists a one-parameter family of developables which corresponds to the reduction of (1) to the Laguerre-Forsyth canonical form. If upon the generator of each of these developables the point be marked where it meets the cuspidal edge, the locus of these points is a cubic curve called the torsal cubic, which has contact of the fourth order with C_y at P_y. This torsal cubic also enters largely into the theory. It coincides with the osculating cubic if the curve belongs to a linear complex.

The covariants may be defined geometrically by introducing the notion due to Halphen of the principal tangent plane of the curve C_y and its osculating cubic. By means of the osculating linear complex this defines a covariant curve C_z on the developable of C_y, and as a consequence also covariant curves C_ρ and C_σ. If, however, the curve belongs to a linear complex all of these curves coincide, so that a special treatment of this case becomes necessary. This is left for a future paper. The author notices, however, a very remarkable theorem for this case. If the curve belongs to a linear complex we may, in an infinity of ways, choose the fundamental tetrahedron so that

four of its edges give rise to developables whose cuspidal edges are described by the four vertices. The other two edges of the tetrahedron will then give rise to ruled surfaces upon each of which the vertices of the tetrahedron trace a pair of asymptotic curves. The latter coincide with the two branches of the complex curve for the derived surface of the second kind. The paper will be offered to the *Transactions* for publication.

2. The general problem of determining the elastic displacements and stresses in a homogeneous spherical shell due to the action of known bodily forces and surface forces has been known as Lamé's problem. A general solution was first given by Lamé, another afterward by Thomson. The application of the general solution to special cases in which the surface and bodily forces are arbitrarily assumed is likely to be laborious. The case of a sphere acted upon by bodily forces due to the attraction of a thin surface layer of matter and by surface forces due to the weight of the layer was discussed by Darwin with reference to the problem of the stresses produced in the earth by the weight of continents. His computations were restricted to surface distributions which could be represented by even zonal harmonics of order not exceeding 12. The present paper by Professor Hoskins refers to the case in which the surface layer is of uniform thickness and bounded by any circle. This distribution is represented by an infinite series of zonal harmonics, each term of which introduces a term into the general solution. The summation of the resulting series is effected and general formulas are found for the stresses at all points in the axis of symmetry of the layer. Numerical results are given for several special cases.

3. Dr. Putnam considers a special case of the problem, to find the factors of a number which belong to a non-homogeneous linear form in any number of variables, the conjugate factor belonging to the same form. Besides the six factors which are always present in $(p^2-1)^2$, he shows that $px+1$ (p being a prime) is a factor if $x=\frac{1}{2}(1+\sqrt{4p+s})$ and $4p+s$ is a square. The latter condition requires that p be of the form $10n\pm 1$. Different statements of the problem were considered and certain other sufficient conditions for additional factors were developed.

4. Major MacMahon's paper was devoted to a study of linear differential operators closely related to his earlier investigations along this line. He considered the operators depending upon $\phi(u) \equiv \phi$ and $\psi(u) \equiv \psi$, where ϕ and ψ are analytic functions capable of expansion in ascending powers of x, and $u = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$. If

$$\phi = \phi(u) = \phi_0 + \phi_1 x + \phi_2 x^2 + \cdots,$$
$$\psi = \psi(u) = \psi_0 + \psi_1 x + \psi_2 x^2 + \cdots,$$

the derived operations may be written

$$(\mu, \nu, n, \phi) = \mu\phi_0\partial_{a_n} + (\mu + \nu)\phi_1\partial_{a_{n+1}} + (\mu + 2\nu)\phi_2\partial_{a_{n+2}} + \cdots,$$
$$(\mu', \nu', n', \psi) = \mu'\psi_0\partial_{a_{n'}} + (\mu' + \nu')\psi_1\partial_{a_{n'+1}} + (\mu' + 2\nu')\psi_2\partial_{a_{n'+2}} + \cdots.$$

It is found that the alternant of two such "functionally derived operators" is expressible as a linear function of operators derived from the three functions

$$\phi\psi', \qquad \phi'\psi, \qquad \int_0^u \phi'\psi'\, du,$$

where the accent denotes differentiation in regard to u.

The condition that the operator derived from

$$\int_0^u \phi'\psi'\, du$$

may be absent is that $\mu/\nu - n = \mu'/\nu' - n' = c$, a constant, and in that case the alternant is an operator derived from the single function $(c + n)\phi\psi' - (c + n')\phi'\psi$.

Designating c as the characteristic of the operator, the author considered the subgroups of operators of constant characteristic. The paper concludes with the following functional group, appertaining to operators which involve a finite number of variables: Let $x^{p+1} + b_1x^p + b_2x^{p-1} + \cdots + b_{p+1} = 0$ be a fixed relation, so that

$$u_p^m = (a_0 + a_1x + a_2x^2 + \cdots + a_px^p)^m$$
$$= A_{m,0} + A_{m,1}x + \cdots + A_{m,p}x^p,$$

and take an operator

$$\Omega_m = A_{m,0}\partial_{a_0} + A_{m,1}\partial_{a_1} + \cdots + A_{m,p}\partial_{a_p}.$$

It is easy to prove that the alternant of Ω_m and Ω_n is expressible in the manner

$$\Omega_m\Omega_n - \Omega_n\Omega_m = (\Omega_m.\Omega_n) = (n-m)\Omega_{n-m+1}.$$

These operators therefore form a transitive group of infinite order involving $p+1$ variables $a_0, a_1, \cdots, a_p$ and $p+1$ parameters $b_1, b_2 \cdots, b_{p+1}$. This group admits in all probability of further generalization.

5. At a previous meeting of the Society, Professor Blichfeldt stated the theorem that the number of parameters of a primitive continuous group in n variables is limited, being less than a number λ which depends only upon n. In the present paper it is shown that if the structure ("Zusammensetzung") of a primitive continuous group G be given, the construction of the infinitesimal transformations of G is reduced to solving a given set of algebraic equations and (possibly) to changing a given set of infinitesimal transformations with algebraic coefficients, generating an intransitive group G' in $n' > n$ variables, into a set generating the required group G, the invariant functions of G' acting as arbitrary constants of G. If the presence of arbitrary constants be allowed in the coefficients of the infinitesimal transformations of a group-type, it follows that the number of types of primitive groups in n variables is limited.

6. The results in Professor Lehmer's paper are well known (Muth, Elementartheiler, 1899, page 217). The methods used are believed to be new and the connection of the invariant figures with the character of the roots of the characteristic quartic is clearly shown.

7. In a recent number of the *Proceedings of the London Mathematical Society* Professor Miller proved the theorem, "The number of the cyclic subgroups of order $p^\alpha(\alpha > 1, p > 2)$ in any group G is of the form kp whenever the Sylow subgroups of order p^m in G are non-cyclic." In the present paper

he supplements this theorem by considering the case when $p = 2$. The main results may be stated as follows: If the Sylow subgroups of order $2^m (m > 1)$ contained in any group G are either cyclic or contain a cyclic subgroup of order 2^{m-1} which includes only two invariant operators under one of these Sylow subgroups, then the number of operators of order 2 in G is of the form $1 + 4k$. When this condition is not satisfied the number of these operators is always of the form $3 + 4k$. When $m = 1$, G contains an invariant subgroup which is composed of all its operators of odd order, and the number of the subgroups of order 2 may have either of the two forms $1 + 4k$, $3 + 4k$. This is the only case where the form of the number of the subgroups of order 2 is not determined by the form of this number in a Sylow subgroup.

G. A. MILLER,
Secretary of the Section.

THE FOUNDATIONS OF MATHEMATICS.*

The Principles of Mathematics. By BERTRAND RUSSELL. Volume I. Cambridge. The University Press, 1903. xxix + 534 pp.

Essai sur les Fondements de la Géométrie. Par BERTRAND RUSSELL. Traduction par A. CADENAT, revue et annotée par l'auteur et par L. COUTURAT. Paris, Gauthier-Villars, 1901. x + 274 pp.

1. *The Problem.* — Pure mathematics has always been conceived in the minds of its votaries and by the world at large to be a science which makes up for whatever it lacks in human interest, and in the stimulus of close contact with the infinite variety of nature, by the sureness, the absolute accuracy, of its methods and results. Yet what has been accepted as sure and accurate in one generation has frequently required fundamental revision in the next. Euclid and his pupils could doubtless have complained of the lack of rigor and logical precision in his predecessors just as forcibly as some modern pupils of Weierstrass berate their scientific ancestors and companions.

* We may also refer our readers to the review by L. Couturat, *Bulletin des Sciences Mathématiques*, vol. 28, pp. 129–147 (1904). So large is the work of Russell that Couturat's review and our own supplement rather than overlap one another.

Euler, finding confusion in the theory of the infinite and infinitesimal, proceeded to explain away the difficulties, that others might be free from the prevailing errors. We cannot accept his reasoning to-day. At the beginning of the last century the state of infinite series was lamentable and Cauchy's memoir on the subject is said to have impressed itself on Laplace to such an extent that he postponed publishing his Mécanique Céleste until he became so hopeless of righting things that he gave up trying to do it. The righting has been accomplished in the present generation by Poincaré. Yet we very much doubt whether Laplace, before hearing of Cauchy's treatment, would have for a moment granted any possible inaccuracy in his own methods. Somewhat later Dirichlet treated the problem of determining a harmonic function from its boundary values and so careful a mathematician as H. Weber extended the method to the discussion of the equation $\Delta V + \lambda V = 0$ without any apparent qualms as to error. Nevertheless, now-a-days, the theoretical importance and the practical use of the principles of Dirichlet and Thomson are completely obscured for many by the too great emphasis laid upon the errors in the original demonstration of the principles.

We notice that the advance toward our present rigor has been made step by step by great men who, however, were no greater — one might almost say no more careful — than their fellows working in apparent unconsciousness of the impending trouble and perhaps even incredulous at first as to its reality. When will this revision stop? And whereunto will it finally lead? This is the problem of the ultimate foundation of mathematics. In attempting an answer one can learn only hesitancy from the past. The delicacy of the question is such that even the greatest mathematicians and philosophers of to-day have made what seem to be substantial slips of judgment and have shown on occasions an astounding ignorance of the essence of the problem which they were discussing. At times this has been due to the inevitable failings of individual intuition in dealing with matters that are still unsettled; but all too frequently it has been the result of a wholly unpardonable disregard of the work already accomplished by others. Even when guarding as much as may be against this latter sin, those who approach the depths of the subject upon which Russell has so courageously entered may well expect to hear the warning:

Procul, o procul este profani!

2. *The Solution.* — Says Russell: Pure mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

This is probably the first attempt to give a complete definition of mathematics solely in terms of the laws of thought and the other necessary paraphernalia of the thinking mind. Some there are who, under the influence of arithmetic tendencies, might be tempted to give a decidedly more superficial definition in terms of integers. Some might regard a complete definition as impossible. The fact that a definition such as the above may be given — and it is the purpose of Russell's Principles of Mathematics to demonstrate that the definition is not illusory nor too small nor too large — is attributable to two things: first, the more careful discrimination of what *pure* mathematics is; second, the extraordinary development of logic since Boole removed it from the trammels of medieval scholasticism.

He to whom the present highly developed state of the foundations of mathematics is chiefly due is Peano — one whose work unfortunately is very little known and still less appreciated in this country. True, Leibniz had long since done much and of recent years has been ably expounded by L. Couturat;* true it is, too, that Boole had freed us from Aristotelianism and that C. S. Peirce† and Schroeder had carried the technique of logic much farther; but they had never accomplished that intimate formal relation between logic and all mathematics which was the necessary precursor to a yet more intimate philosophic relation and which has been brought about by Peano aided by a large school of pupils and fellow-workers. The advance has been made largely by introducing into sym-

* La Logique de Leibniz, Paris, 1901.

† The fundamental importance of the logic of relations (see infra) was emphasized by C. S. Peirce in 1880-1884: but it is only beginning to have its full effect.

bolic logic such a simplification of notation as to relieve it of its unwieldiness and to allow its development into a powerful instrument without which one can hardly hope to get the best results in the treacherous though treasure-laden fields of the foundations of mathematics. Poincaré, to be sure, in his review of Hilbert's Foundations of Geometry * spurns this *pasigraphy*, characterizing it as disastrous in teaching, hurtful to mental development, and deadening for investigators, nipping their originality in the bud. However much we may agree in the first statements (see § 7, page 91), we had best be cautious in accepting such sweeping statements as the last, even from so great an authority — especially in view of the fact that, equipped with this pasigraphy, the Italian investigators, Peano and his pupil Pieri,† with some rights of priority, had given a more fundamental *logical* ‡ treatment of the subject on

* Translated in BULLETIN, vol. 10. p. 5 (Oct., 1903).

† "I principii della geometria di posizione," *Memorie della R. Accademia delle Scienze di Torino*, vol. 48, pp. 1–62. And, "Della geometria elementare come sistema ipotetico-deduttivo; Monografia del punto e moto," *ibid.*, vol. 49, pp. 173–222.

‡ While we appreciate and admire as much as anyone can the beauties of Hilbert's famous Grundlagen der Geometrie, we fail to see how the historical facts can justify what Poincaré says (*l. c.*, p. 23): "He has made the philosophy of mathematics take a long step in advance, comparable to those which were due to Lobachevsky, to Riemann, to Helmholtz, and to Lie." Poincaré makes the point that Hilbert regards his geometric elements as *mere things* and on this seems to rest a large part of the praise (*l. c.* bottom p. 21 and top p. 22). If this be so, it ought to be mentioned as a matter of history that Peano, in 1889, in his Principii di Geometria took precisely this stand, p. 24. In 1891–2, Vailati, *Rivista*, vol. 1, p. 127, vol. 2, p. 71, again formulated the principle in words. By 1897 the Italian school had gone as far beyond this point of view as to make it a *postulate* that points are classes — thus showing a twofold advance, once in recognizing the presence of a postulate, again in using the word class so as to bring the reasoning into form dependent upon precise logical processes alone. It has also been said that the idea of the independence of the axioms was due to Hilbert. As a matter of fact in 1894, Peano, "Sui fondamenti della geometria," *Rivista*, vol. 4, pp. 51 et seq., states the problem and, by actually setting up simple systems of elements, proves the independence of certain axioms from certain others. So that by 1899 the idea and method were both five years old at least. Again, in 1889, Peano laid it down as a principle that there should be as few undefined symbols as possible, and he used but few. In 1897–9 Pieri used but two for projective geometry and but two for metric geometry, whereas Hilbert was using a considerable number, seven or eight. (The idea of compatibility seems to have been first stated clearly by Hilbert.) There still remains in the Grundlagen der Geometrie matter enough for the amplest praise. The archimedean axiom, the theorems of Pascal and Desargues, the analysis of segments and areas, and a host of things are treated either for the first time or in a new way, and with consummate skill. We should say that it was in the technique rather than in the philosophy of geometry that Hilbert created an epoch.

which Poincaré was writing than is to be found in the work he was praising so highly. In the fields of arithmetic and algebra, too, Burali-Forti and Padoa, adherents of Peano, had reached a point far beyond the widest view of the chief of the German school that deals with the same subjects.* Furthermore, on this one point Poincaré may not be regarded as an authority; for his own work † in the field should be characterized as subjective rather than objective, speculative and suggestive rather than purely logical.‡ Anyone who is acquainted with the articles presented to the Philosophical Congress at Paris in 1900 by Peano, Burali-Forti, Padoa, and Pieri, cannot be convinced that these authors had become deadened, and the artificiality of their system is by no means so certain as it might be. Since then, our author, Russell, has simplified and improved the older work of C. S. Peirce on the theory of relations, adapting it to the system of Peano, and has produced a coherent treatment of the great problems underlying mathematics. In view of accomplished facts one inclines more readily to the praise given by Whitehead: "I believe the invention of the Peano and Russell symbolism forms an epoch in mathematical reasoning." §

3. *The Reason.* — It is not hard to detect the reason why mathematics has thus pushed its foundations back until they

* Compare the papers below referred to in the Bibliothèque du Congrès International with No. 2 of Hilbert's Mathematical Problems, BULLETIN, vol. 8, p. 447. We may refer also to Padoa, *L'Enseignement Mathématique*, vol. 5, p. 85. See also § 4 of the present review. For our readers, who may be working on the problem No. 2, we may note — what we unfortunately failed to note at the time of translating — namely, that a solution along the lines proposed by Hilbert seems logically impossible. A solution has long since been proposed in the article here referred to. There are those, however, who hold that Padoa has gone so far as to overshoot the mark. Hilbert has again taken up the matter much more searchingly than in 1900. It is to be regretted that his paper which was presented at Heidelberg, August, 1904, is not at hand for comparison.

† La science et l'hypothèse, Paris, 1903; and numerous scattered essays.

‡ That Poincaré seems frequently to have in mind the physical rather than the mathematical, the psychological rather than the logical point of view can be seen in several places in his review. On p. 8 he asserts that we know the axioms are non-contradictory "since geometry exists." And on p. 22 he seems to complain that the logical standpoint interests the author to the utter disregard of the psychological. It should be remembered that the first chief aim of the modern researches on the foundations of geometry is to be entirely rid of the psychological element — and this for the very reason that secondly we may decide just what that psychological element must be. This latter problem belongs rather to the philosopher and psychologist than to the mathematician.

§ *American Journal of Mathematics*, vol. 24, p. 367 (October, 1902).

have come to rest solely in logic. In the first place mathematical or other reasoning presupposes a mind capable of rational, that is, non-selfcontradictory ratiocinative processes. Now it always has been comfortably assumed that we can carry out such processes if only we are careful enough, that there is no need of formulating the laws of thought before beginning to reason, or even that a formulation and analysis of those laws is impossible.* Where then do the errors creep in? An examination of some typical cases shows that it is generally through lack of a sufficiently careful definition of the terms. This failure properly to define has led to interminable discussions which from the start could only lead either to nothing or to wrong results. In mathematics it is the absence of precise definition which brings in the erroneous statements concerning differentiation, continuity, and infinity, with a host of others. The perception of this difficulty was the origin of the principle of arithmetization and of epsilon proofs. In the end, however, after one has really mastered the principles of modern analysis he seldom needs the actual presence of epsilons to establish a theorem. Nevertheless it is a satisfaction to have this formal method to fall back on whenever challenged by one's own hesitancy or by that of others. In like manner, who has not at times during some long complicated or indirect logical demonstration felt the least bit uncertain; who would not be glad to have at his hand some formal method such as Peano's, based upon certain rudimentary propositions and concepts?

In truth it is a matter of more consequence than is sometimes thought, to have clearly in mind those processes which are definitely to be admitted as logical. The one process which stands out most definite in our consciousness is the syllogism. If a piece of reasoning can be put in the form of major premise, minor premise, conclusion, we are tolerably sure of its truth. But numerous proofs cannot be so constructed and it is one of the most frequent errors committed by the intuitive logicians to say that reasoning consists in a sequence of syllogisms. Perhaps the greatest advance made by Boole was the clear recognition of the necessity of asyllogistic reasoning.

The question then becomes of fundamental importance: What is at the bottom of our logic? When the matter is

* Duhamel, Des méthodes dans les sciences, Paris, 1875, vol. 1, p. 17.

looked into, it appears that we constantly use *propositions*, passing from certain propositions as hypothesis to certain others as conclusion. The laws of implication which govern the relation between hypothesis and conclusion constitute the logical theory or calculus of propositions. Casting about for other principles we come upon *classes* or sets of objects represented in ordinary speech by common nouns. The development of the interrelations of classes produces the logical calculus of classes. This calculus has a remarkable analogy with the calculus of propositions, but the relation is not quite dual. In the third place we perceive that *relations* are of the utmost importance. Every transformation, every function is a relation. In common language the verb does but express a relation between the subject and object. Thus there appears the necessity for a calculus of relations.* The complete logical calculus, as now used, is a combination of these three types. The whole number of laws of thought or logical premises which seem to be required for establishing the calculus in all the generality necessary for mathematics is small. In addition to these premises there are a certain number of elementary ideas or terms such as implication, and the notions of proposition, class and relation, which must be assumed as known. It is the discussion of these questions which are of a philosophical rather than mathematical nature, that fills the first Part of Russell's Principles.

We may grant, then, that logic is *necessary* to mathematics. It is affirmed to be *sufficient*. This in reality is the remarkable content of the definition given by the author. So immune are we from logical error that the necessity of logic might never force us to a critical examination of its principles; but the affirmation of its sufficiency fully justifies and even renders imperative such an examination. Russell's entire volume is devoted to establishing this sufficiency. And although the subject is very new and many difficulties philosophical and mathematical are still outstanding, there can be little doubt that to an unexpectedly large extent the author is successful in his attempt and that in these Principles he has given a per-

* Peano and his immediate followers overlook the importance of this subject — so busy are they with other important questions. It is one of the lasting services of Russell, following very closely on the work done twenty years earlier by C. S. Peirce, to have recognized the necessity of this addition to Peano's system and to have supplied the deficiency. See his articles in the *Revue de Mathématiques*, vol. 7, nos. 2 and following (1901–1902).

manent set to the future philosophy of the questions which he handles.

4. *Some Notions.* — Owing to the wide-spread diversity of usage in the meaning of such fundamental notions as postulates, axioms, undefined symbols, definitions, consistency, independence (of postulates), irreducibility (of undefined symbols), completeness (of systems of postulates and undefined symbols), we think it best to enter upon some slight exposition * of these matters instead of taking up the critical discussion of some of the more abstruse problems which are treated by the author and which could scarcely be appreciated before such exposition.

Axiom is a word which has so long been used in so many vague ways that its use in pure mathematics had probably best be abandoned. The familiar definition: An axiom is a self-evident truth, means, if it means anything, that the proposition which we call an axiom has been approved of by us in the light of our experience and intuition. In this sense pure mathematics has no axioms: for mathematics is a formal subject over which formal and not material implication reigns.† The proper word to use for those statements which we posit would seem to be postulate. What self-evident truths can there be concerning objects which are not dependent on any definite interpretation but are merely marks to be operated upon in accordance with the rules of formal logic? Postulates, however, may be laid down at will so long as they are not contradictory. It is the postulates which give the objects their intellectual though not physical existence. Indeed before we can apply to the physical world any of the systems of logical geometry, for instance, we have the one great axiom: This system fits nature sufficiently for our purposes. To postulate such a statement would avail us naught. We must carefully consider the totality of our experience and decide whether the statement seems to represent a truth.

Definition is a term which has long been used by philosophers to stand for a process of analysis and exemplification

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*See also E. V. Huntington on "Sets of independent postulates for the algebra of logic," *Transactions*, vol. 5, p. 288 (July, 1904).

†In regard to logic on which mathematics rests, we should incline to use the word axiom (if indeed we do not prefer to hold to premise) and not postulate. For here we are dealing with the actual (mental) world and not with a system of marks. The basis of rationality must go deeper than a mere set of marks and postulates. It is foundation of everything and must be more *real* than anything else.

which brings before the mind a real consciousness of the object defined. This sort of definition has to be used in dictionaries. In mathematics, however, no such vague process is permissible. Mathematical definition is simply the attributing of a name to some object whose existence has been established or momentarily postulated. It is the process of replacing a set of statements by a single name and is resorted to solely for convenience. In any science whose development has been perfected, definitions may be entirely done away with by those who are willing to sacrifice brevity. There can be little doubt that a large number of definitions might better be thus put out of the way.*

Although all definitions are thus merely nominal, there are three distinct aspects † of definition which are worth considering in detail in connection with the theory of integers. These may be characterized as (1) the particular definition, (2) the definition by postulates, (3) the definition by abstraction. They may be illustrated as follows: Suppose (1) that it is possible to find a logical class K of which the elements are, let us say, classes or propositions. Suppose further that by means of logical processes alone we may define operations on the elements like addition and multiplication of integers. Grant that there exists in the class K an element analogous to zero (in case K is a class of classes this would be the null-class; in case K were a class of propositions it would be the absurd). In fact suppose that we could set up a class K and a set of operations in K which have the properties of integers as we use them. We then might say from a formal point of view that the class K was the class of integers, that the elements of K were the integers themselves, and that the operations we had set up were the ordinary operations of arithmetic. This would be a satisfactory though very particular definition of the integers and would have the advantage that unless there were a contradiction inherent in our logic there could be no contradiction in our system of integers. Or (2) we may assume a certain set of elementary terms, known as undefined symbols, such as num-

* Peano: Bibliothèque du Congrès International de Philosophie à Paris, 1900, vol. 3 (1901).

† Compare Burali-Forti; Bibliothèque etc. "Sur les différentes définitions du nombre réel." We say *aspect* on account of a change in view which has been established since 1900. Also "Le classe finite" *Atti della Accademia reale di Scienze di Torino*, vol. 32, p. 34 (1896); and more recently "Sulla teoria generale delle grandezze e dei numeri," *ibid.*, vol. 39 (Jan., 1904).

ber, zero, and successor. These we could connect, as Peano and Padoa have done, by a system of postulates, and thus we should have a definition of number through postulates. In order to prove the non-contradictoriness of our postulates and indefinables, that is, the existence of our elements, we should have to set up some system which afforded one interpretation of the indefinables and of the postulates. As this must be done by going back to the laws of thought we finally get very near to where we started in the other sort of definition. The definition remains, however, slightly more general: for the integers thus defined are not merely one set of elements but any set which satisfies the postulates. Or (3) we may use the principle of abstraction on which Russell places a great deal of emphasis. We may say that two classes of objects, no matter what objects they be, have the same number when there exists a one-to-one relation between their elements.* Thus number becomes the common property of all similar classes, and is their only common property. The class of numbers becomes the class of all similar classes. Owing to Russell's development of the theory of relations this definition becomes also merely nominal and as it seems to be the most fundamental and philosophic it may be accepted as the best thus far given.

Although the use of postulates other than the premises of logic, and the use of undefined symbols other than those of logic seem needless and to be avoided in pure mathematics, the usage is so common that we may go on to say a few words concerning consistency, independence, irreducibility, and completeness — especially as these ideas are somewhat usable in the foundations of logic. To show the consistency of the system of postulates and undefined symbols it is evidently futile to attempt to develop the consequences of the postulates until no contradiction is reached (this method of stating the thing is sufficient to show wherein lies the futility): for the most that can be accomplished in this way is to see that up to a certain point no contradiction has been reached. The method of proof consists merely in finding some system of entities known to exist and affording a possible interpretation of the undefined symbols and postulates. To make the proof really fundamental for the system of integers it seems necessary to go quite out of the

* Russell shows, Principles, p. 113, that this idea is not dependent on the general concept number, nor even on the concept unity. Two classes which can be placed in one-to-one correspondence are called *similar*.

field of mathematics into the domain of logic.* The method of showing the independence is merely to set up for each postulate one existent system of elements in which there are possible interpretations of our undefined symbols and which satisfies all the other postulates but not the particular one in question. This shows the independence of that one. If one of the undefined symbols used in the statement of the postulates can be given a nominal definition in terms of the others the system of indefinables is redundant. It was Padoa † who first made effective use of this idea. To show the irreducibility of the indefinables relative to the system of postulates it is necessary to set up a system of elements which satisfies all the postulates, which affords an interpretation of the undefined symbols, and which continues to satisfy these conditions when one of the undefined symbols is suitably altered: this must be done for each. The problem is quite similar to that of the independence of the postulates and is not difficult to solve in case the number of undefined symbols is small. All this difficulty is avoided in dealing with the different branches of mathematics when Russell's point of view — no new indefinables, no new postulates — is taken.

Huntington ‡ seems to have been the first to bring to effective use the idea of completeness. The problem is to show that if there are two sets M and M' of objects each § of which satisfies the postulates and affords interpretations of the indefinables, then the two sets of objects may be brought into one-to-one correspondence in such a way as to preserve the interpretation of the symbols. With the statement of this last idea we have arrived at the limit of present ideas concerning the interrelations of the notions at the base of mathematics as defined by postulates.

* See references given in footnote under § 2, p. 77. The consistency is far more important than the independence, irreducibility, or completeness: for these are merely a matter of elegance, whereas that determines whether or not all our reasoning upon the system in question is void.

† Bibliothèque etc. "Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque." This remarkable essay should be read by every one. We may note that Padoa uses 'transformateurs' but introduces no theory of relations. In this respect Russell has introduced improvements.

‡ *Transactions*, vol. 3, pp. 264–282 (1902). See also Veblen, *Transactions*, vol. 5, p. 346 (1904).

§ Serious mistakes, resulting in definitions of no essential content, have been made by forgetting that the relations which connect the elements must be in correspondence, in addition to any correspondence between the elements. See also footnote under § 6, p. 87.

5. *Numbers.* — The analysis of number, cardinal or ordinal, finite or infinite, integral, rational, or real, with carefully drawn distinctions between the many allied ideas such as counting, quantity, magnitude, and distance, forms the content of Parts II–V., pages 111–370, of the Principles. To do anything like justice to this masterpiece of analysis in a field so strewn with difficulties would be impossible within the space at our command. The summaries given by the author at the close of each Part afford a clear review of the ideas which have been discussed and the points which have been won. Leaving out of account the advances which are made toward the precision of the terms which lie at the bottom of logic we can at best merely indicate some of the results which are of greatest interest to mathematicians.

It is shown that cardinal and ordinal integral numbers are inherently different, that finite cardinals and ordinals may be obtained in terms the one of the other but that this principle cannot be applied to the infinite. With the guidance of the principle of abstraction cardinal integer has been defined as a class of similar classes. This definition has the immediate advantage of giving finite and infinite cardinals at the same time. The finite may then be distinguished from the infinite by the fact that in the former the whole cannot be similar to its part, whereas in the latter it can. Another point which Russell establishes with the aid of Whitehead * is that by the use of logical addition the numerical addition of a finite or infinite number of finite or infinite cardinals may be and indeed (if we invoke the principle of abstraction) should be defined in such a manner that the order in which the numbers are added plays no part. This is a great victory for common sense and must appeal to everyone as a vindication of the school-child in his inherent notion that he has the same number of marbles whether he has five in one pocket and three in another or three in two pockets and two in a third, no matter which of his pockets these be. The principle of commutation and association of the terms in addition is entirely done away with, except in so far as mechanical difficulties prevent us from writing simultaneously a number of terms and the signs of addition connecting them. We may point to the fact that the work applies equally well to finite and infinite sums as an indication of its extreme generality

* *American Journal of Mathematics*, vol. 24 (1902).

and as evidence that at last we have a principle of addition distinctly above the plane of counting on one's fingers. In like manner the definition of multiplication is such as to be free from the laws of commutation and association of the factors and to apply equally to finite or infinite products of finite or infinite cardinals. Again a vindication of the school-child who rightly cannot see why it should make any difference whether he puts down four rows of three marks or three rows of four.

The discussion of the meaning of quantity and magnitude in Part III. and its connection with number we will not pause to consider, but we pass directly to the theory of order as developed in Part IV. The treatment of this subject is greatly simplified by the theory of relations. Order is shown to be an asymmetric transitive relation, an essential property of serial relations. It is clearly pointed out and it is important to notice that when a set of objects is given the relation is not necessarily included; whereas when the relation is given the field in which it operates must also be given. If recourse is had to the principle of abstraction the ordinal integer appears as "the common property of classes of serial relations which generate ordinally similar series." As the cardinals are classes of similar classes, so the ordinals are classes of like relations. The principle of induction is intimately associated with the system of ordinals rather than with the system of cardinals although for finite numbers the distinction is not so great as for infinites. We may say that the finite ordinal is that which can be reached by induction from 1. It appears that those who generate their system of numbers by a relation of succession or by counting — that is, by successive acts of attention — must in reality be coming at something which resembles ordinals much more nearly than cardinals. The difference between the theory of infinite cardinals and infinite ordinals brings to light the important fact that in mathematics we have two kinds of infinite: the cardinal, which has the property of being similar to a part of itself, and the ordinal, which cannot be reached by induction from 1.* The discussion naturally brings up the old question of extensive and intensive definition. The definition of an

* This would seem to render invalid the contention of Poincaré in his La science et l'hypothèse to the effect that the principle of induction is the essence of the infinite. We have seen that it is the essence of the finite. The difficulty seems to be that Poincaré has in mind the definition of the infinite as a growing variable. If this be so, the apparent contradiction resolves itself into a mere difference of definitions.

object is said to be extensive when the object is given by the enumeration of its parts; it is said to be intensive when the object is characterized by its properties. In the treatment of these questions and of transfinite cardinals and ordinals there is much which is instructive for the mathematician and the philosopher. The author points out with his customary frank desire to state no more than the truth that there still remain difficulties to solve. Thanks to his lucid and modest presentation there is no reason why he should not find adherents who will take up the work and attempt the solution in a spirit of hearty coöperation.

There is a school of creationists who, when they find that certain infinite processes lead to no rational limit nor yet to a number which becomes infinite, postulate the existence of a limit and thus obtain the irrational numbers. The author does not consider an *ipse dixit* like this to be a sufficiently good theorem of existence. He therefore considers infinite sets of rationals and by means of them he forms a set of things which he calls real numbers. A real number is neither a rational nor an irrational; it is a certain infinite set of rationals. The real numbers thus defined are shown to satisfy the notion of a continuum. According to the method followed, the continuum appears as an idea wholly ordinal in nature. With the aids thus prepared the author is able to give a very satisfactory account of the philosophy of the infinite and of the continuous. His treatment of the paradoxes of Zeno shows that the arguments of the ancient philosopher are by no means so far from right as might be imagined and that the contradictions are more apparent than real.

6. *Geometry and Mechanics.* — A short study of the properties of multiple series leads to a point from which it may be seen that: Geometry is the study of series of two or more dimensions.* In this manner the necessity of new postulates and new indefinables is avoided. The procedure is evidently reasonable. Mathematical geometry has long since been divested of all spatial relations between its elements. The above definition

* As the serial relation is emphasized rather than its domain (see discussion of order given above) the author avoids a definition which is null and which makes dimensions impossible. Compare discussion of completeness and footnote, § 4. For a fuller discussion of this important point we may refer to "The so-called foundations of geometry," by the present reviewer, in the *Archiv der Mathematik und Physik*, vol. 6, pp. 104–122 (1903). Toward the end of the discussion a change, which may cause some confusion, is made to the point of view of physical geometry. The first part, however, deals solely with purely mathematical geometry.

is but the culmination of the ideas of manifolds introduced by Grassmann and Riemann. As those who define geometry by postulates are forced to show the existence of their elements by having recourse to systems of numbers the question is quite pertinent: Why not begin with a purely nominal definition like the above and avoid the trouble of proofs of existence, of independence, and of irreducibility?

At this juncture it is interesting to compare the attitude taken in the Principles with that taken in the older Foundations of Geometry. It should be remembered that the author originally started with the study of the philosophy of dynamics and hence necessarily of geometry. To render the examination really searching the foundations of geometry had to be investigated. But, once started, the end was not so easily to be reached. Probing into the mysteries of infinity and continuity led to arithmetic in its wider sense. Trying to render precise the meaning of important words such as element, set, operation, conclusion, proof, etc., could but conduct to the study of logic, and the desire to be rid so far as possible from the contaminations of the personal element brought up at last at formal logic. And then the entire field had to be traversed in the forward direction with the necessity of constant acquisition of original results at every step! Surely the present work is a monument to patience, perseverance, and thoroughness.

In the essay on the foundations of geometry the author had not yet reached the logical stage — scarcely the arithmetic stage. He was content, as some still are, to analyse the ideas in the rough, to use a large number of indefinables, to state broad indefinite axioms instead of brief incisive postulates — in a word, to forego all the modern technique. The result was an extremely suggestive essay — one which still can be read with profit and by rights ought to be read, if only for the sake of contrast, in connection with the newer work. To-day we have mathematical geometry, then we had physical. If one wishes to read an excellent account of space from the physico-metaphysical point of view he has but to turn to the Russell of a few years since; if he would know the extreme point of modern mathematico-logical geometry he has merely to take up the Russell of to-day.

In Part VII. the analysis proceeds to mechanics. Here space is merely a certain three- (or n-) dimensional series; time, a simple series. There is a relation which connects part of space

(the material points) with all time, that is, $a, b, c = R(a_0, b_0, c_0, t)$, where a, b, c are the coördinates of the material points. This relation R is so chosen as to allow for the impossibility of generating or destroying matter. The relation is also chosen so that if the relation between matter and time is known at two instants it is known at every instant. In this way is stated the causality in the universe. This seems very far off from the real world. It must delight the hearts of philosophers who believe in a pure idealism. It is found that arithmetic may be handled adequately with no help save from logic. This does not surprise us. Then geometry is put in the same category. Modern mathematicians have so accustomed us to look on merely the logical side of the subject that we are not troubled. Finally comes dynamics. Why not thermodynamics, electrodynamics, biodynamics, anything we please? There is no reason why not. There is in reality no place to stop, save when we have become tired of pure logic, if once we include geometry. As a matter of fact all our concepts whether of space, or matter, or electricity, or life, are but idealizations more or less well-defined, and, if we insist on subjecting the world to purely logical explanation, they all belong in the same class.

Upon this matter we may best quote Russell who, amid all his refinements, keeps a clear idea of their proper place in the system of all knowledge. He says: The laws of motion, like the axiom of parallels in regard to space, may be viewed either as parts of a definition of a class of possible material universes, or as empirically verified assertions concerning the actual material universe. But in no way can they be taken as *à priori* truths necessarily applicable to any possible material world. The *à priori* truths involved in dynamics are only those of logic; as a system of deductive reasoning, dynamics requires nothing further, while as a science of what exists, it requires experiment and observation. Those who have admitted a similar conclusion in geometry are not likely to question it here; but it is important to establish separately every instance of the principle that knowledge as to what exists is never derivable from general philosophical considerations, but is always and wholly empirical. *

* It would be interesting to discuss in how far this attitude is really in accord or out of accord with the apparently very different view of Poincaré (La science et l'hypothèse) that the question whether the parallel axiom is true or not true is devoid of sense owing to the fact that it is merely a *convenient* method of correlating experience and a *convention* can have neither truth nor falsity.

7. *Some Conclusions.*—There is one conclusion in logic which suggests itself almost inevitably at this point. For there are a considerable number of systems of logic current at present. Different authors have treated the subject differently — each choosing the system of indefinables and laws of thought which seemed best to him at the time. Now it is by no means true that these various systems of logic have been proved coextensive or even not mutually contradictory. If it should appear that they cannot be brought into harmonious relation one with another there will be some instructive, if bewildering, conclusions to draw. And as we have such complex entities as infinity and continua with which to deal it might not be regarded as surprising if some points were found to stand out permanently, so that logicians will permanently disagree. In fact at present there seems to be a grave logical difficulty in our logical system as developed by Russell. This trouble had been felt by Frege and a solution had been proposed by him; but it does not seem entirely satisfactory.* In view of the outstanding difficulties and the possible divergence of systems of logic held by equally good authorities, we come to the conclusion that it is dangerous to accept the naïve point of view of those who claim that a certain piece of reasoning depends on the operation of logic alone but who fail to state what those operations are and to use all the means possible to avoid the intrusion of extraneous ideas. They may not fall into error, but they are merely following in the footsteps of those who "knew" what infinity and continuity were.

From the pedagogic point of view we may also draw some conclusions. It is hardly necessary to trouble the student with the commutative and associative laws in multiplication of integers or with elaborate deductions of a number system before he is readily able to appreciate the needlessness of the former and the relation which the latter bears to the theory of finite and infinite cardinals and ordinals, the ideas of compactness and continuity, and the two kinds of infinity.

* In a long appendix, Russell gives a detailed exposition of the important work of Frege, which culminated in the Grundgesetze der Arithmetik, and he discusses this troublesome contradiction again from a different standpoint. It is this contradiction which Hilbert had in mind in his Heidelberg address referred to under § 2. He, therefore, attempts to recast the principles of logic and of arithmetic in such a manner as to render them sufficient for mathematical reasoning. We certainly hope that he has succeeded in doing so to the satisfaction of both mathematicians and philosophers.

A clear-cut physical conception that numbers possess order and may be associated with the points on a line is a workable idea which in practice is both necessary and sufficient for ordinary rigorous analysis. An inadequate vague idea regarded as a useful working hypothesis seems, on the whole, productive of more good and less harm than an inadequate definite idea regarded as final. In geometry and mechanics the physical attitude may be taken. Axioms, things deemed worthy of credence on the basis of experience, should take the place of postulates. This does not prevent, in fact it encourages, the statement of a large number of axioms without troubling too much as to their independence. At the same time these statements should include the essential idea of order and the useful idea of continuity and other ideas which are usually passed over.* In short we should use and train intuition to the utmost in connection with some logic; for pure logic alone is, as Poincaré states (§ 2), harmful to the earlier development of the mind.

From the mathematical standpoint we have learned that many of the objects which have been thought of as individual must be regarded as classes. We cannot define euclidean space, but we can define the class of all euclidean spaces.† The principle of abstraction, here involved, seems to arise from the necessity of taking the terms in a logical equation to represent the common attribute of all the objects which may in some way satisfy the equation. As during the progress of the discussion, we have introduced no new indefinables, no new postulates, no processes other than those of logic, there is no possibility of our arriving at contradictions except through the failure of our logical system to be logical; and behind this we cannot go. It remains merely to show the existence of the classes with which we have dealt; otherwise our work would be null. To quote freely from our author: The existence of zero is derived from the fact that the null-class is a member of zero; the

* Compare, for example, the series of articles by the reviewer on Spherical Geometry, *American Mathematical Monthly*, commencing January, 1904.

† This apparently considerably lowers the importance of the idea of *completeness* discussed in § 4. For it appears as if the one-to-one correspondence between the different euclidean spaces were really of minor significance. This is but another instance of the fact that the elements themselves are unimportant — that it is the abstraction from them which is most fundamental. However, we believe that the idea of completeness is a new step, a step onward and toward a fuller description of the systems dealt with.

existence of 1, from the fact that zero is a unit-class (the null-class being its only member). By an evident induction we get all finite numbers. From the class of finite cardinals follows first the existence of the smallest of the infinite cardinals, and second, by considering them in the order of magnitude, the existence of ordinals and the smallest of the infinite ordinals. We may go on to obtain the rationals, compact enumerable series, continuous series. From the last we may see the existence of complex numbers, of the class of euclidean spaces, of projective spaces, of hyperbolic spaces, and of spaces with various metrical properties. Finally we may prove the existence of the class of dynamical worlds. Throughout this process no entities are employed but such as are definable in terms of the fundamental logical constants. Thus the chain of definitions and existence-theorems is complete, and the purely logical nature of mathematics is established throughout.

This is as far as we are conducted. But we are promised a second volume — may it be soon forthcoming — written with the collaboration of Whitehead. Herein will be contained actual chains of deduction leading from the premises of logic through arithmetic to geometry. Herein will also be found various original developments in which the notations of Peano and Russell have been found useful. For those who wish sooner to get at the Peano-Russell point of view in the matter, we append a bibliography, which while very incomplete may still be found useful in tracing the development of the ideas:

(1) Arithmetices principia nova methodo exposita, Turin, Bocca Frères, 1889.

(2) I principii di geometria logicamente esposti, Turin, 1889.

These two works by Peano are the starting point of the whole movement. They were written in the days when a careful explanation and translation of the symbolic method was in vogue and form a good starting point for study. The *Formulaire de Mathématiques*, edited by Peano, is rather hard to begin on. The *Rivista di Matematica*, now the *Revue de Mathématiques*, also edited by Peano, furnishes much easy and instructive reading matter. Logica matematica by Burali-Forti in the series of Manuali Hoepli may serve as a textbook. Omitting important memoirs by Burali-Forti on arithmetic and by Pieri on geometry which we have quoted in footnotes, we cite again

(3) Bibliothèque du congrès international de philosophie, volume 3 (1901).

The articles by Peano, Burali-Forti, Padoa, and Pieri show the point at which the Italian school had arrived in 1900. It is since that time that most of Russell's technical work has appeared. For the present state of the science, we would note a memoir by Whitehead:

(4) "On cardinal numbers," *American Journal of Mathematics*, volume 24 (1902), pages 367–394; and a still more recent paper by Burali-Forti, "Sulla teoria generale delle grandezze e dei numeri," *Atti della R. Accademia delle Scienze di Torino*, volume 39, (January, 1904).

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YALE UNIVERSITY,
July, 1904.

NOTES.

THE closing (October) number of volume 26 of the *American Journal of Mathematics* contains the following papers: "Invariants of a system of linear partial differential equations, and the theory of congruences of rays," by E. J. WILCZYNSKI; "On elements connected each to each by one or the other of two reciprocal relations," by C. DE POLIGNAC.

THE opening (October) number of volume 6 of the *Annals of Mathematics* contains the following papers: "On the subgroups of an abelian group," by G. A. MILLER; "Note on the continued product of the operators of any group of finite order," by W. B. FITE; "Reduction of an elliptic integral to Legendre's normal form," by N. R. WILSON; "The necessary and sufficient condition under which two linear homogeneous differential equations have integrals in common," by A. B. PIERCE; "A general method of evaluating determinants," by G. MACLOSKIE; "Application of groups to a complex problem in arrangements," by L. E. DICKSON; "On functions defined by an infinite series of analytic functions of a complex variable," by M. B. PORTER.

AT the Cambridge meeting of the British association for the advancement of science (cf. October BULLETIN, page 28), Professor A. R. FORSYTH presided over the subsection of pure mathematics, whose programme included the following papers: "A fragment of elementary mathematics," "Geometry of the complex variable," by Professor F. MORLEY; "Peano's

symbolic method," by Mr. A. N. WHITEHEAD; "The theory of linear partial differential equations," by Major P. A. MACMAHON; "The roots of the characteristic equation of a linear substitution," by Professor T. J. I'A BROMWICH; "The zeros of Taylor's series," by Mr. G. H. HARDY; "Practical trigonometry deduced from a lemma of Archimedes on the trisection of an angle," by Professor P. MANSION; "Binary canon extension," by Lt. Colonel A. CUNNINGHAM; 'On the theory of transfinite numbers," by Dr. E. W. HOBSON; "Report on the theory of point groups," "Notes on plane curves," by Mr. H. HILTON; "Note on a special homographic transformation of a screw system," by Professor Sir R. S. BALL; "Note on the theory of continuous groups," by Professor A. R. FORSYTH; "The aims of geometry," by Professor W. F. MEYER; "The theory of vibrations," by Professor V. VOLTERRA; "On the stability of the steady motion of a viscous fluid," by Mr. W. McF. ORR; "The laws of error," by Professor F. Y. EDGEWORTH; "Note on the Schwarzian derivative," by Professor A. C. DIXON; "Some observations on linear difference equations," by Rev. E. W. BARNES; "On the use of divergent series in astronomy," by Mr. Z. U. AHMAD.

THE annual meeting of the French association for the advancement of science was held at Grenoble, August 4–11, under the presidency of C. A. LAISANT. M. CH. ANDRÉ, director of the observatory at Lyons, presided over the section of mathematics. Besides several papers on astronomy, the following mathematical papers were read: "On resolvent arithmetic spaces," by Professor ARNOUX; "Choice of principles for instruction in geometry," by M. COMBEBIAC; "Reflections concerning non-euclidean hypotheses," by General FROLOV; "Enumeration of finite and infinite continuous groups in space of three dimensions," by Professor LE VASSEUR; "On the number of prime numbers between zero and n," by Professor LEBON; "Construction of magic squares," by M. TARRY; "Discussion of Professor Méray's methods of instruction in geometry." At the close of this discussion it was voted to invite the minister of public instruction to encourage the introduction of the new method into the secondary schools and the preparation of manuals for the use of the pupils.

The next meeting of the association will be held at Cherbourg, with Professor N. GIARD as president, and Professor G. SAUGRAIN as secretary.

AT the International congress of mathematicians at Heidelberg, there was circulated a printed list of questions, prepared by the editors of *L'Enseignement Mathématique* in collaboration with the Geneva psychologists, Professor TH. FLOURNOY and Dr. E. CLAPARÈDE, on the methods of work practiced by mathematicians. The questions are divided into three groups. The first group deals with the psychological side, embracing such topics as the genesis of a taste for the science, influence of heredity, relative interest in pure and applied mathematics, the roles of chance and inspiration, unconscious cerebration, practice of assimilating existing literature as preparation for an investigation or of leaving the mind at first unfettered, advice to beginners. The second group relates to the mathematician's mode of life, hours of work, regularity of habits, avocations, physical exercise, requisite vacations. The third group invites miscellaneous data of value for the inquiry, effect of narcotics, favorite attitude and appliances while at work, nature of mental images, personal recollections of the habits of deceased mathematicians, etc. Communications respecting the inquiry may be addressed to Professor H. FEHR, 19 rue Gevray, Geneva, Switzerland.

THE second International congress of philosophy met in Geneva, September 4–8, 1904. Of the five sections two were of particular interest to mathematicians, one on the philosophy of the sciences and the other the international congress of the history of the sciences. In the former section the following papers upon mathematics were presented: By Dr. PIERRE BOUTROUX of Paris: "On the notion of correspondence in mathematical analysis"; by Professor JULES ANDRADE of Besançon: "Mechanical geometry"; by Dr. ARNOLD REYMOND of Lausanne: "On geometric judgment."

The section on the history of the sciences, which was occupied almost exclusively with mathematics at the first congress held in Paris in 1900, divided the time this year so as to give considerable attention to the physical sciences. The section was organized by M. PAUL TANNERY, as president of the international commission appointed by the Congress of the history of the sciences, held in Rome in 1903. The papers of special interest to mathematicians were as follows: By Professor H. G. ZEUTHEN of Copenhagen: "The origins of scientific geometry (the Pythagorean theorem)"; by Professor VICTOR

MORTET of Paris: "On geometry among the Latins"; by M. HENRY BERR of Paris: "Gassendi as an historian of the sciences"; by Baron CARRA DE VAUX: "On the mechanics of the Arabs"; by Professor P. DUHEM of Bordeaux: "History of dynamics (on the acceleration produced by a constant force)."

In Professor Zeuthen's paper, read by M. Tannery, the attempt is made to discover not the method by which Pythagoras demonstrated his theorem (a probably futile inquiry), but the succession of ideas leading to the earlier discovery of the proposition by the Hindus and the Egyptians. His analysis of the processes in the Çulvasutras has led to certain unexpected points of contact and to new conjectures which merit serious attention.

Professor Duhem, by an argument based upon undisputed texts, established a thesis already suggested by M. Paul Tannery, that the discovery of Galileo's law was inspired by ideas essentially different from those of to-day. The preamble of every history of dynamics must henceforth be this: The notion of force in dynamics is a concept created by Newton and defined by him through the enunciation of principles now classic. Before Newton's time the word was used in a vague and very different sense; the only clear mechanical notions were those of movement and speed.

The congress was hospitably entertained by the citizens of Geneva. On September 4 a reception was given at the residence of Dr. E. Claparède, the general secretary. On Monday evening, September 5, a steamboat excursion was taken to Coppet, followed by a dinner and by the illumination of the lake front. On Tuesday evening there was an excursion to Mont Salève, and on Wednesday evening one to Chougny, where a reception was held by M. Agénor Boissier. On Thursday evening the congress closed with a banquet given by the city of Geneva, in the foyer of the theater.

AN official history of the Deutche Mathematiker-Vereinigung, prepared by the secretary, Professor A. GUTZMER, of Jena, was issued on the occasion of the recent Heidelberg Congress. Besides the history of the society's development, the pamphlet contains a report of each of its annual meetings, the list of members, and the table of contents of each volume of the *Jahresbericht*.

On the occasion of the Heidelberg Congress, the firm of B. G. Teubner in Leipzig issued the hundreth edition of its catalogue of mathematical and technical books. Among the books in advanced preparation may be mentioned the following: Clebsch-Lindemann, Vorlesungen über Geometrie, Teil 1, second edition; Enriques, Fragen der Elementargeometrie, translated into German by H. Fleischer; Fricke, Kurzgefasste Vorlesungen über verschiedene Gebiete der höheren Mathematik, Teil 2: Algebra und Geometrie; Minkowski, Geometrie der Zahlen, zweite Lieferung; Weber und Wellstein, Encyklopädie der Elementar-Mathematik, volumes 2 and 3.

Arrangements have been completed for extending Cantor's Geschichte der Mathematik by the addition of one or possibly two more volumes, covering the period from 1758 to 1799. The task will be undertaken by the following collaborators, and is to be completed in 1906: 1. History, classical editions, lexicons, S. Günther of Munich; 2. Textbooks on elementary geometry, theory of parallels, V. Bobynin of Moscow; 3. Trigonometry and tables, A. von Braunmühl of Munich; 4. Algebra and theory of numbers, F. Cajori of Colorado; 5. Series, combinations, probabilities and imaginaries, E. Netto of Giessen; 6. Analytic geometry of the plane and of space, V. Kommerell of Reutlingen; Descriptive geometry, G. Loria of Genoa; 7. Textbooks on the calculus, definite integrals and transcendents, G. Vivanti of Messina; 8. Total and partial differential equations, calculus of variations and finite differences, C. R. Wallner of Munich; 9. Development of mathematics from 1759 to 1799, M. Cantor of Heidelberg.

The Open Court publishing company of Chicago has begun the publication of a series of portaits of mathematicians, from the collection of Professor D. E. Smith. The first installment of twelve, chosen to represent important mathematical advances prior to 1700, is now in the press.

The various foreign universities will offer courses in mathematics during the winter semester of 1904–1905 as follows:

University of Berlin. — By Professor H. A. Schwarz: Differential calculus, four hours; with exercises, two hours; Theory of elliptic functions, four hours; Selected chapters on analytic functions, two hours; Colloquium, two hours; Semi-

nar, two hours. — By Professor G. FROBENIUS: Algebra, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Theory of functions, four hours; Analytic mechanics, four hours; Seminar, two hours. — By Professor J. KNOBLAUCH: Definite integrals, four hours; Theory of surfaces, four hours; Twisted curves, one hour. — By Professor G. HETTNER: Introduction to the theory of determinants, two hours. — By Professor R. LEHMANN-FILHÉS: Analytic geometry, four hours. — By Dr. E. LANDAU: Integral calculus, four hours; with exercises, one hour; Theory of quadric surfaces, two hours; Theory of aggregates, one hour. — By Dr. J. SCHUR: Algebraic theory of quadratic forms, two hours: Theory of numbers, four hours.

UNIVERSITY OF BONN. — By Professor H. KORTUM: Algebra, four hours; Quadratic forms, two hours; Seminar, one hour. — By Professor E. STUDY: Mechanics I, three hours; Analytic geometry II, four hours; with exercises, one hour; Seminar, one hour. — By Professor L. HEFFTER: Differential and integral calculus II, four hours; Descriptive geometry with exercises, five hours; Seminar, one hour.

UNIVERSITY OF KÖNIGSBERG. — By Professor F. W. MEYER: Applied mathematics, four hours; with exercises, one hour; Determinants, two hours. — By Professor A. SCHOENFLIES: Introduction to the theory of differential equations, four hours; Quadric surfaces, two hours; Seminar, one hour. — By Professor L. SAALSCHÜTZ: Bernoulli's numbers, two hours; Integral calculus, four hours; with exercises, two hours. — By Dr. T. VAHLEN: Non-euclidean geometry, three hours.

UNIVERSITY OF STRASSBURG. — By Professor T. REYE: Geometry of position, three hours; Analytic mechanics, two hours; Seminar, one hour. — By Professor H. WEBER: Differential and integral calculus, four hours; Algebra, three hours; Seminar, one and one half hours. — By Professor M. SIMON; History of mathematics, two hours. — By Professor M. DISTELI: Plane analytic geometry, three hours; Descriptive geometry with exercises, four hours; Theory of screws, one hour; Seminar, one hour. — By Professor J. WELLSTEIN: Differential and integral calculus, four hours. — By Dr. P. EPSTEIN: Introduction to the theory of numbers, three hours.

UNIVERSITY OF CZERNOWITZ. — By Professor K. DAUBLEBSKY VON STERNECK: Differential and integral calculus, five hours; Seminar, two hours.

UNIVERSITY OF GRAZ. — By Professor J. FRISCHAUF: Algebraic analysis, three hours; Analytic geometry, two hours. — By Professor K. V. VON DANTSCHER: Analysis, five hours; Seminar, two hours. — By Professor H. STREINTZ: Descriptive geometry, three hours; Statistics I, three hours. — By Dr. K. CARDA: Partial differential equations of the first order, two hours. — By Dr. J. PLEMELJ: Linear differential equations, two hours. — By Dr. J. GRÜNWALD: Line geometry, two hours.

UNIVERSITY OF GENOA. — By Professor G. LORIA: Theory of algebraïc forms with applications to geometry, four hours. — By Professor F. PORRO: Theoretical astronomy, four hours. — By Professor O. TEDOME: Calculus of variations and differential equations, four hours. — By Dr. E. ALMANSI: The Helmholtz theory of vortices, four hours.

GERMAN UNIVERSITY OF PRAGUE. — By Professor G. PICK: Differential and integral calculus, five hours; Seminar, two hours. — By Professar J. GMEINER: Algebraic equations, three hours; Theory of functions of a complex variable, two hours.

UNIVERSITY OF VIENNA. — By Professor G. VON ESCHERICH: Theory of functions, five hours; Seminar, two hours. — By Professor MERTENS: Algebra, five hours; Seminar, two hours. — By Professor W. WIRTINGER: Differential and integral calculus, five hours; with exercises, two hours; Seminar, two hours. — By Professor G. KOHN: Synthetic geometry, four hours; with exercises, two hours; Differential geometry, two hours. — By Dr. A. TAUBER: Mathematics of insurance, four hours. — By Dr. E. BLASCHKE: Mathematical theory of statistics, two hours.

PROFESSOR M. BRENDEL, of Göttingen, has been elected a member of the Leopold-Carolus academy of natural sciences.

PROFESSOR R. FRICKE has been elected rector of the technical school at Brunswick, and a corresponding member of the Göttingen royal society.

PROFESSOR H. GRASSMANN, of Halle, has been appointed associate professor of mathematics at the University of Giessen; he was also recently elected a member of the Leopold-Carolus academy of natural sciences.

DR. F. GUARDUCCI, of Florence, has been appointed professor of geodesy at the University of Bologna.

PROFESSOR L. HEFFTER, of Bonn, has been appointed professor of mathematics at the technical school at Aachen.

PROFESSOR A. PRINGSHEIM, of Munich, has been elected corresponding member of the Göttingen royal society.

PROFESSOR K. ROHN, of Dresden, has been appointed professor of mathematics at the University of Leipzig, and is to transfer from Dresden at Easter, 1905.

DR. K. SCHREBER has been appointed professor of mechanics at the University of Greifswald.

PROFESSOR J. SOMMER, of Poppelsdorf, has been appointed professor of mathematics at the new technical school at Danzig.

PROFESSOR L. SYLOW, of Christiana, has been named knight of the order "pour le mérite" for arts and sciences.

M. J. TANNERY has been appointed professor of the calculus in the faculty of sciences at the University of Paris.

PROFESSOR A. WANGERIN, of Halle, has been elected a corresponding member of the Academy of sciences at Erfurt.

DR. G. HERGLOTZ has been appointed docent in mathematics and astronomy at the University of Göttingen.

PROFESSORS O. E. MEYER, of Breslau and F. ROTH, of Strassburg, have retired from giving academic courses.

PROFESSOR F. LINDEMANN has been elected rector of the University of Munich for the year 1904–1905. Professor F. DINGELDEY has been elected rector of the technical school at Darmstadt. Professor F. SCHUR has been elected rector of the technical school at Karlsruhe.

DR. G. PRASAD has been appointed professor of mathematics at the Muir Central College in Allahabad, India.

AT the University of California, Dr. Dr. D. N. LEHMER and Mr. A. W. WHITNEY have been promoted to assistant professorships of mathematics. Dr. B. L. NEWKIRK has been appointed instructor, and Mr. A. J. CHAMPREUX assistant in mathematics. Dr. E. M. BLAKE has been appointed professor of mathematics at the University of Arizona, succeeding Professor S. M. WOODWARD, who has accepted a position in the engineering department of the University of Iowa.

PROFESSOR H. V. GUMMERE, of Ursinus College, has been appointed professor of mathematics in the Drexel Institute, Philadelphia.

THE following academic appointments are also announced: Dr. E. L. DODD, instructor in mathematics, University of Iowa; Mr. A. HALL, instructor in mathematics, U. S. Naval Academy; Mr. J. H. BARTON, instructor in mathematics, Dartmouth College; Dr. H. C. CONVERSE, instructor in mathematics, Baltimore Polytechnic Institute; Mr. W. W. DAVIS, tutor in mathematics, Northwestern University; Messrs. T. E. GRAVATT and C. F. SHOOP, instructors in mathematics, Pennsylvania State College.

AT Rutgers College, Professor E. A. BOWSER has been made emeritus professor of mathematics.

MR. R. W. H. T. HUDSON, lecturer in mathematics in the University of Liverpool, met with a fatal accident in the mountains of north Wales on September 2. Mr. Hudson, who was the son of Professor W. H. H. Hudson, of King's College, London, was but twenty-eight years of age.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMSON (E.). Ueber eine zwei-zweideutige Punktverwandtschaft der Ebene und eine ein-eindeutige, welche mit ihr in Zusammenhang steht. (Diss., Erlangen.) München, Straub, 1904. 8vo. 46 pp., 1 plate. M. 0.75

ARENDT (G.). See LEJEUNE-DIRICHLET (G.).

BLICHFELDT (H. F.). On the order of linear homogeneous groups (second paper). 4to. (*Transactions of the American Mathematical Society*, vol. 5, pp. 310–325.)

BÖHMER (P.). Ueber geometrische Approximationen. (Diss.) Göttingen, 1904. 8vo. 55 pp., 2 plates. M. 1.60

BUNGERS (E.). Ueber das Cylindroid. (Progr.) Sagan, 1904. 4to. 30 pp.

BURKHARDT (H.). See ENCYKLOPÄDIE and JAHRESBERICHT.

CAJORI (F.). An introduction to the modern theory of equations. New York, Macmillan, 1904. 8vo. 9 + 239 pp. Cloth. $1.75

CAMPBELL (D. F.). The elements of the differential and integral calculus, with numerous examples. New York, Macmillan, 1904. 12mo. 12 + 364 pp. Cloth. $1.90

DIRICHLET. See LEJEUNE-DIRICHLET (G.).

DYCK (W. VON). See ENCYKLOPÄDIE.

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. (In 7 Bänden.) Vol. I: Arithmetik und Algebra, redigiert von W. F. Meyer. Heft 8: W. von Dyck, Einleitender Bericht über das Unternehmen der Herausgabe der Encyklopädie der mathematischen Wissenschaften; W. F. Meyer, Vorrede und Register zu Band I; Inhaltsverzeichnis zu Band I (Teil 1 und 2); A. Pringsheim, Unendliche Prozesse mit komplexen Termen. Leipzig, Teubner, 1904. 8vo. 38 + 10 pp., pp. 1121–1197.

——. Vol. II: Analysis, redigiert von H. Burkhardt. Heft 5: A. Sommerfeld, Randwertaufgaben in der Theorie der partiellen Differentialgleichungen; A. Kneser, Variationsrechnung; E. Zermelo und H. Hahn, Weiterentwickelung der Variationsrechnung in den letzten Jahren; H. Burkhardt, Trigonometrische Interpolation (mathematische Behandlung periodischer Naturerscheinungen); A. Wangerin, Theorie der Kugelfunktionen und der verwandten Funktionen, insbesondere der Lamé'schen und Bessel'schen (Theorie spezieller, durch lineare Differentialgleichungen definierter Funktionen). Leipzig, Teubner, 1904. 8vo. Pp. 561–759.

FORDERMANN (A.). Ueber die Zahlformen, deren Quadratwurzel eine gegebene Kettenbruchperiode liefert. (Progr.) Wilmersdorf-Berlin, 1904. 4to. 21 pp.

FREITAG (W.). Diskussion des dreifach orthogonalen Flächensystems, dessen eine Schar in Cartesischen Koordinaten durch die Gleichung $x^m y^n z^p = c$ dargestellt wird. (Progr.) Torgau, 1904. 4to. 37 pp.

GRASSMANN (R.). Die Ausdehnungslehre oder die Wissenschaft von den extensiven Grössen in strenger Formelentwicklung. Nebst Formelbuch. Stettin, 1904. 8vo. 9 + 132 + 14 pp. M. 1.50

GREENHILL (A. G.). The third elliptic integral and the ellipsotomic problem. 4to. (*Philosophical Transactions of the Royal Society of London*, series A, Vol. 203, pp. 217–304.)

GRÜNINGER (W.). Das System der Kegelschnitte mit drei festen Punkten und einer festen Tangente. (Diss.) Giessen, 1903. 8vo. 52 pp.

HAHN (H.). See ENCYKLOPÄDIE.

HALSTED (G. B.). The Lobachevski prize. (*Science*, new series, 20, pp. 353–367.)

HEEGAARD (P.). Foredrag over integralregningen. Kjöbenhavn, 1904. 4to. 152 pp. (Autogr.) M. 1.50

HERBERICH (G.). Eine neue Klasse von reellen algebraischen Raumkurven konstanter Torsion. (Progr.) München, 1904. 8vo. 23 pp.

HOYER. Ueber arithmetische Bestimmung der endlichen Gruppen. (Progr.) Burg, 1904. 4to. 12 pp.

JAHRESBERICHT der Deutschen Mathematiker-Vereinigung. Vol. 10, Heft II: Entwicklungen nach oscillirenden Functionen; Bericht, erstattet der Deutschen Mathematiker-Vereinigung von H. Burkhardt. 4te Lieferung. Leipzig, Teubner, 1904. 8vo. Pp. 769–1072.

KARPINSKI (L. C.). Ueber die Verteilung der quadratischen Reste. 4to. (*Journal für die reine und angewandte Mathematik* 127, pp. 1–19.)

KNESER (A.). See ENCYKLOPÄDIE.

KOCH (W.). Ueber die Anwendung der Doppelintegrale in der Funktionentheorie. (Progr.) Sorau. 1904. 4to. 11 pp.

KOENIGSBERGER (L.). Carl Gustav Jacob Jacobi. Festschrift zur Feier der 100. Wiederkehr seines Geburtstages. Mit einem Bildnis und dem Faksimile eines Briefes. Leipzig, Teubner, 1904. 8vo. 18 + 554 pp. Cloth. M. 16.00

——. Carl Gustav Jacob Jacobi. Rede zu der vom internationalen Mathematiker-Kongress in Heidelberg veranstalteten Feier der hundertsten Wiederkehr seines Geburtstages gehalten am 9. August 1904. Mit einem Bildnis C. G. J. Jacobis. Leipzig, Teubner, 1904.

KÖLMEL (F.). Ableitung der verschiedenen Formen der ebenen Kurven dritter Ordnung durch Projektion und Klassifikation derselben. III: Die Kurven vom Geschlechte eins ohne Oval. (Progr.) Baden-Baden, 1904. 4to. 14 pp.

LEJEUNE-DIRICHLET (G.). Vorlesungen über die Lehre von den einfachen und mehrfachen bestimmten Integralen, herausgegeben von G. Arendt. Braunschweig, Vieweg, 1904. 8vo. 23 + 476 pp. M. 12.00

MEYER (W. F.). See ENCYKLOPÄDIE.

MÜLLER (J. O.). Ueber die Minimaleigenschaft der Kugel. (Diss.) Göttingen, 1903. 8vo. 51 pp.

NOTH (G.). Differentialinvarianten und invariante Differentialgleichungen zweier zehngliedriger Gruppen. (Diss.) Leipzig, 1904. 8vo. 32 pp.

PORTER (M. B.). On functions defined by an infinite series of analytic functions of a complex variable. 4to. (*Annals of mathematics* (2) 6, pp. 45–48.)

PRINGSHEIM (A.). See ENCYKLOPÄDIE.

RADFORD (E. M.). Mathematical problem papers, compiled and arranged. London and New York, Macmillan, 1904. 16mo. 6 + 203 pp. Cloth. $1.50

SCHMITT (A.). Ueber involutorische Transformationen. (Diss.) Freiburg, 1904. 8vo. 31 pp.

SÉGUIER (J. A. DE). Théorie des groupes finis. Eléments de la théorie des groupes abstraits. Paris, Gauthier-Villars, 1904. 8vo. 2 + 180 pp. Fr. 5.00

SOMMERFELD (A.). See ENCYKLOPÄDIE.

SONNET (H.) et FRONTÉRA (G.). Eléments de géométrie analytique, rédigés conformément au programme d'admission à l'Ecole polytechnique et à l'Ecole normale supérieure. 10e édition. Paris, Hachette, 1904. 8vo. 758 pp., plates. Fr. 8.00

STURM (A.). Geschichte der Mathematik. Leipzig, Göschen, 1904. 16mo. 152 pp. Cloth. (Sammlung Göschen, No. 226.) M. 0.80

WANGERIN (A.). See ENCYKLOPÄDIE.

ZERMELO (E.). See ENCYKLOPÄDIE.

II. ELEMENTARY MATHEMATICS.

ALEY (R. J.). The essentials of algebra; for secondary schools. New York, Silver, Burdett & Co. [1904]. 12mo. 5 + 295 pp. Cloth. (Standard series of mathematics.) $1.00

BAKER (W. M.) and BOURNE (A. A.). Elementary algebra. Part I. London, Bell, 1904. 12mo. 336 pp. Cloth. 2s. 6d.

——. Examples in algebra, selected from Elementary algebra. London, Bell, 1904. 12mo. Cloth. 1s. 6d.

BARBETTE (E.). Cours de trigonométrie. Gand, 1904. 8vo. 260 pp. Fr. 5.00

BERTRAND (J.) et GARCET (H.). Traité d'algèbre. Première partie, à l'usage des classes de mathématiques élémentaires. 18e édition. Paris, Hachette, 1904. 8vo. 4 + 330 pp. Fr. 5.00

BORCHARDT (W. G.) and PERROTT (A. D.). New trigonometry for schools. Part I. London, Bell, 1904. 12mo. 278 pp. Cloth. 2s. 6d.

BÖTTGER (A.). Die ebene Geometrie. Für den Unterricht an der Realschule bearbeitet. 4te, nach den neuesten Lehrplänen bearbeitete Auflage. Leipzig, Dürr, 1904. 8vo. 160 pp. Cloth. M. 1.80

BOULTON (E. S.). Geometry on modern lines for elementary students. London, 1904. 8vo. 134 pp. 2s.

BOURNE (A. A.). See BAKER (W. M.).

BUDDEN (E.). Elementary pure geometry, with mensuration. Parts 1 and 2. London, Chambers, 1904. 8vo. 200 pp. Cloth. 2s.

——. Elementary pure geometry, with mensuration; a complete course of geometry for schools. London, Chambers, 1904. 8vo. 292 pp. Cloth. 3s.

CARTON. Eléments de géométrie, adaptés au programme de baccalauréat de l'enseignement secondaire (section A et B). 6e édition. Paris, Poussielgue, 1904. 16mo. 316 pp. (Alliance des maisons d'éducation chrétienne.)

COMBEROUSSE (C. DE). See ROUCHÉ (E.).

DEAKIN (R.). Euclid, books 1–6. Certificate edition. With papers set at certificate examinations during the last 5 years. London, Clive, 1904. 12mo. 358 pp. Cloth. 2s. 6d.

DUMONT (F.). Introduction à la géométrie du troisième ordre. Annecy, Dépollier, 1904. 8vo. 9 + 317 pp.

FOURREY (E.). Récréations arithmétiques. 3e édition. Paris, Vuibert et Nony, 1904. 8vo. 8 + 263 pp.

GARCET (H.). See BERTRAND (J.).

GÖTTING (E.). See KLEIN (F.).

GRANDA (J.). Tratado de algebra elemental, con numerosas aplicaciones, conforme con el programa oficial para segunda enseñanza, aprobado por el Consejo superior de instrucción pública, dedicado á los alumnos de los colegios de segunda enseñanza, á la Escuela militar y á los aspirantes á la Escuela especial de ingenieros de construcciones civiles y de minas. 3a edición, corregida y aumentada. Lima, 1904. 4to. 298 + 6 pp. Fr. 2.00

——. Tratado de algebra práctica, acompañado de numerosos ejercicios y problemas resueltos y por resolver. Lima, 1902. 4to. 110 + 2 pp. Fr. 1.50

——. Tratado de geometría elemental, con numerosas aplicaciones conforme con el programa oficial para instrucción media, aprobado por el Consejo superior de instrucción pública, dedicado á los alumnos de los colegios de segunda enseñanza y á los aspirantes

de la Escuela especial de ingenieros de construcciones civiles y de minas. 2a edición, corregida y aumentada. Lima, 1903. 4to. 282 + 6 pp., 11 plates. F. 2.00

——. Tratado de trigonometría, con numerosas aplicaciones, aprobado por el Consejo superior de instrucción pública, dedicado á los alumnos de los colegios de instrucción media, á la Escuela militar y á los aspirantes á la Escuela especial de ingenieros de construcciones civiles y de minas. 2a edición, corregida y aumentada. Lima, 1901. 4to. 136 + 4 pp. Fr. 1.50

HESSENBERG (G.). Ebene und sphärische Trigonometrie. 2te, verbesserte Auflage. Leipzig, 1904. 8vo. 167 pp. M. 0.80

JACQUET (E.) et LACLEF (A.). Eléments d'algèbre; exercices et problèmes, à l'usage des écoles primaires supérieures, des écoles professionnelles, des écoles normales d'instituteurs. Paris, Nathan, 1904. 12mo. 174 pp.

KLEIN (F.). Ueber eine zeitgemässe Umgestaltung des mathematischen Unterrichts an den höheren Schulen. Vorträge. Mit einem Abdruck verschiedener einschlägiger Aufsätze von E. Götting und F. Klein. Leipzig, Teubner, 1904. 8vo. 4 + 82 pp. M. 1.60

LACLEF (A.). See JACQUET (E.).

MENGER (J.). Grundlehren der Geometrie. Ein Leitfaden für den Unterricht in der Geometrie und im geometrischen Zeichnen an Realschulen. Mit vielen Konstruktions- und Rechnungsaufgaben. 7te Auflage. Inhaltlich unveränderter nach der neuen Rechtschreibung hergestellter Abdruck der 6ten Auflage. Wien, Hölder, 1904. 8vo. 4 + 127 pp. Cloth. M. 1.65

PERROTT (A. D.). See BORCHARDT (W. G.).

PONDICHY (A.). Le postulatum d'Euclide est-il démonstrable? Solution et théorie des parallèles. Bruxelles, Falk, 1904. 8vo. 32 pp. Fr. 1.50

ROUCHÉ (E.) et COMBEROUSSE (C. DE). Eléments de géométrie. Classes de mathématiques A et B. 7e édition, conforme aux programmes du 31 mai 1902, revue et complétée par E. Rouché. Paris, Gauthier-Villars, 1904. 8vo. 41 + 652 pp. Fr. 6.00

SCHOCH (W.). Introduction to geometry; a manual of exercises for beginners. Boston, Allyn & Bacon, 1904. 12mo. 5 + 137. Cloth. $0.60

SCHULTZ (E.). Leitfaden der Planimetrie für gewerbliche Lehranstalten. Teil 2. 3te Auflage. Essen, Baedeker, 1904. 8vo. 4 + 94 pp. Boards. M. 1.00

SCHUMANN (E.). Lehrbuch der ebenen Geometrie für die ersten drei Jahre geometrischen Unterrichts an höheren Schulen. Stuttgart, Grub, 1904. 8vo. 9 + 202 pp. Cloth. M. 2.20

SITTIG. Ableitung von sin $(\alpha + \beta)$ und cos $(\alpha + \beta)$ am Dreieck. (Progr.) Coburg, 1904. 4to. 10 pp.

SOLUTIONS raisonnées des questions de géométrie proposées dans le cours des écoles régimentaires, à l'usage des sous-officiers candidats aux écoles militaires de Saint-Maixent, Saumur, Versailles, Vincennes. 3e édition. Paris, Lavauzelle [1904]. 16mo. 156 pp. Fr. 3.00

SUÁREZ CHIGLIONE (A.). Elementos de geometría, con arreglo al nuevo programa oficial, tomados de la obra de matemáticas elementales. Valencia [1904]. 8vo. 143 pp.

TANNER (J. H.). Answers to Tanner's Elementary algebra. New York, American Book Co. [1904]. 12mo. 23 pp. $0.10

TESTI (G. M.). Corso di matematiche ad uso delle scuole secondarie superiori e più specialmente degli istituti tecnici. Vol. 6: Trigonometria piana e sferica. Livorno, Giusti, 1904. 8vo.

TEYSSENNEAU (E.). Cent récréations mathématiques; curiosités scientifiques. Paris, 1904. 16mo. 185 pp. Fr. 6.20

WOLF (F. C.). Praktische Geometrie für den Schul- und Selbstunterricht. Nach den Grundsätzen der Anschauung und Konzentration in genetischer Stufenfolge aufgebaut und unter besonderer Berücksichtigung der praktischen Bedürfnisse bearbeitet. Heft 1 und 2. 2te durchgesehene Auflage. Leipzig, Wunderlich, 1904. 8vo. 81 pp. M. 0.80

III. APPLIED MATHEMATICS.

ALLÉVI (L.). Théorie générale du mouvement varié de l'eau dans les tuyaux de conduite. Paris, 1904. 4to. 48 pp. M. 1.80

AMES (J. S.). Text-book of general physics, for high schools and colleges. New York, American Book Co. [1904]. 8vo. 768 pp. Cloth. $3.50

ANCEY (C.). Théorie des opérations d'assurance. Paris, Rousseau, 1904. 8vo. 8 + 470 pp.

ANTOMARI (X.). Traité de géométrie descriptive, à l'usage des élèves des classes de mathématiques, des aspirants au baccalauréat, et des candidats à l'Institut agronomique et à l'Ecole navale. 3e édition. Paris, Vuibert et Nony, 1904. 8vo. 164 pp.

AVERLY (A.). Le problème général du "vol" et la force centrifuge. Fascicule I: Principes fondamentaux de la mécanique; du mouvement dans l'atmosphère; travail intégralement nécessité. Paris, Dunod, 1904. 8vo. 96 pp. Fr. 3.00

BERNIOLLE (P.). Cours de géométrie descriptive, conforme au programme du 11 août 1902 pour la préparation à l'Ecole militaire de Saint-Cyr. Paris, Paulin, 1904. 16mo. 301 pp. Cloth. Fr. 4.00

BOBROVSKY (S. P.). Statics. Part I: Statics of the rigid system; theory of centers of gravity; theory of resistance to motion; simple machines. With examples and exercises. St. Petersburg, 1904. 8vo. 300 pp. (Russian.) M. 9.00

BRADSHAW (J. W.). Ueber die Flächendichtigkeit der Elektricität auf unendlich langen Cylindern. (Diss.) Strassburg i. E., Goeller, 1904. 8vo. 33 pp., 1 plate.

BREYDEL (A.). Nature intime de l'électricité, du magnétisme et des radiations. Paris, Dunod, 1904. 8vo. 100 pp. Fr. 3.50

BUCHANAN (R.). The mathematical theory of eclipses according to Chauvenet's transformation of Bessel's method explained and illustrated, to which are appended transits of Mercury and Venus and occultations of fixed stars. Philadelphia, Lippincott, 1904. 8vo. 10 + 17 + 247 pp. Cloth. $5.00

CHAMPION (E.). Principes généraux d'électricité théorique et pratique. Vol. I: Thermodynamique; électrostatique; propriétés fondamentales du courant continu; électrolyse, piles. Lille, 1904. 4to. 207 pp.

CHEVRIER (G.). Etude sur les résonnances dans les réseaux de distribution par courants alternatifs. Paris 1904. 8vo. 76 pp. Fr. 3.00

DAVENPORT (C. B.). Statistical methods, with special reference to biological variation. 2d edition, enlarged. New York, Wiley, 1904. 12mo. 8 + 223 pp. Morocco. $1.50

DINTER (A.). Definitive Bahnbestimmung des Kometen 1888 V. (Diss.) Breslau, 1903. 4to. 31 pp.

DREYSSÉ (A.). Instruction détaillée sur la règle à calcul Mannheim et méthode pratique, à l'usage de MM. les ingénieurs, architectes, conducteurs de travaux, élèves des écoles du gouvernement, des classes de mathématiques spéciales et de mathématiques élémentaires, etc. Paris, Vuibert et Nony [1904]. 8vo. 160 pp.

DUROY DE BRUIGNAC (A.). Remarques sur la stabilité des bateaux à hélice dans le roulis et les girations. Paris, Béranger, 1904. 8vo. 56 pp.

FERRARIS (G.). Opere publicate per cura della associazione elettroteenica italiana. Vol. 3 (ultimo). Milano, Hoepli, 1904. 8vo.

GERBER (P.). Ueber den Einfluss der Bewegung der Körper auf die Fortpflanzung der Wirkungen im Aether. (Progr.) Stargard, 1904. 4to. 28 pp.

GORODENSKY (M.). On the question of the influence of the earth's rotation on the disturbances of the atmosphere. St. Petersburg, 1904. 8vo. 97 pp., 1 map. (Russian.) R. 1.60

GUICHARD (C.). Traité de mécanique. Première partie: cinématique, à l'usage des élèves des classes de première C et D. 2e édition. Paris, Vuibert et Nony, 1904. 8vo. 8 + 108 pp.

HAESEN (E.). Cours de balistique intérieure. Bruxelles, 1904. 8vo. 257 pp. Fr. 6.00

HEINZERLING (F.). Dreieck und Kraftübertragung in Baukonstruktionslehre und Bauwesen. Grundzüge einer Dynamo-Statik der Baugefüge. Leipzig, Scholtze, 1904. 8vo. 11 + 96 pp., 3 plates. M. 5.50

HELM (G.). Die Theorien der Elektrodynamik nach ihrer geschichtlichen Entwickelung. Leipzig, Veit, 1904. 8vo. 8 + 164 pp. M. 5.60

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